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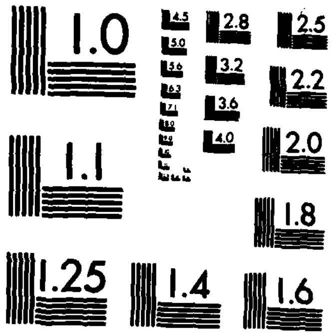

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OPTIMALITY CRITERION METHODS IN STRUCTURAL OPTIMIZATION

N. S. Khot

Analysis & Optimization Branch
Structures & Dynamics Division

October 1982

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Optimization algorithms based on an optimality criterion to design a minimum weight structure are presented. The algorithms are derived for the direct design variable and the reciprocal design variable, and their relationship is discussed. The use of different algorithms and their effect on the convergence behavior is illustrated with sample problems. The presentation is limited to structures which can be analyzed by the finite element method and which are subjected to the constraints on stresses, displacements, minimum and maximum sizes and system stability.

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FOREWORD

This report is prepared as part of an in-house effort under Project 2401, Task No. 240102, "Design and Analysis Methods for Aerospace Vehicle Structures," and Work Unit 24010244, "Analysis and Optimization of Aerospace Structures." The work was carried out in the Design and Analysis Methods Group of the Analysis and Optimization Branch (FIBR), Structures and Dynamics Division, Flight Dynamics Laboratory, Air Force Wright Aeronautical Labs., Wright-Patterson AFB, Ohio.

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SECTION I

INTRODUCTION

This report covers methods based on an optimality criterion to design a minimum weight structure. These methods can be called indirect methods, since the objective is to obtain a design that satisfies a certain specified criterion and by so doing indirectly minimize the weight of the structure. The criterion may be intuitive or derived mathematically based on the nature of the problem. We will concentrate on a mathematical criterion derived by differentiating the Lagrangian with respect to the design variables. In the optimality criterion methods the criterion is derived for the dominant type of constraint imposed on the structure, and that criterion is used to develop the algorithm. The methods are iterative in nature because of the nonlinearity of the constraints and the statical indeterminacy of the structure. In deriving the optimality criterion and developing the algorithm, full use is made of the knowledge of the behavior of the constraints imposed on the structure. The algorithms are efficient because they are specifically developed for structural optimization and generally treat the constraints that weakly affect the behavior of the structure as passive constraints.

The early discussion on the optimality criterion methods as applied to a discretized structure for different types of constraints may be found in References 1 through 4. The application of the method to the stress-displacement-constrained problem, with a variation in the recurrence relations and the methods used to estimate the Lagrange multipliers is discussed in References 5 through 20. The general instability of the structure as a constraint on the structure is considered in References 21 and 22. The optimization of a structure with dynamic loads is discussed in References 23 through 25. These are a few of the references in the general field of structural optimization based on the optimality criterion approach. Additional references may be found in the recent survey paper (Reference 26), and a detailed review of the state-of-the-art in Reference 27. In Reference 28 a comparison and the relationship between the different optimality criterion based algorithms are presented. The relationship between the optimality criterion and mathematical programming methods is presented in Reference 29. The relationship between the structural

algorithms based on an optimality criterion and the projection method is discussed in References 30 and 31. Some of the computer programs based on the optimality criterion to design minimum weight structures are documented in References 32 through 36.

The constraints imposed on the structure may include the maximum allowable stress in each element, the displacement limits at one or more locations, system stability, dynamic stiffness, local element buckling etc. In addition to these there would be limitations on the minimum and maximum sizes of the elements. "An optimality criterion can be derived that includes all these constraints, and it may be desirable to find a design that satisfies this criterion." However, to develop an efficient algorithm based on such a criterion and effectively handle all types of constraints would be impractical and generally unnecessary. For most problems it is developing the algorithm that is more difficult than deriving the optimality criterion. In the case of most structures it is likely that one can predict the type of constraint which will be the most active at the optimum and use the algorithm based on that constraint. Then one can treat all other constraints as passive constraints. It is highly unlikely that all types of constraints will be active at the optimum. Sometimes this point of view may not be correct and will not give us an absolute minimum weight design. However, it will give us a near minimum weight design, and the corresponding optimization algorithm will be efficient and easy to use for a structure with a large number of design variables. Even with this approximation to the overall problem, when the total number of constraints of the same type are large, as we will see, it is advantageous to make additional approximations to reduce the computational effort.

The optimality criterion derived for all the constraints imposed on the structure is equivalent to the Kuhn-Tucker conditions of nonlinear mathematical programming. However, in deriving the optimality criterion and the corresponding structural optimization algorithm, some of the constraints are treated as side constraints in order to simplify the algorithm. A good example of this is the minimum and maximum size limits on the design variables. These constraints generally are not included in the constraint equations and do not enter the optimality criterion. The optimality criterion derived for structural optimization for a particular type of

constraint gives information on the distribution of energy in the structure necessary to have a minimum weight design. The nature of the energy depends upon the type of constraint.

The optimization procedure may be divided into two major steps. These are (1) the analysis of the structure, and (2) the redistribution of the material so that the weight of the structure is reduced and the active constraints are satisfied. These two steps are repeated alternately until a satisfactory design is obtained. The analysis of the discretized structure is performed by the finite element method. The redistribution of the material is carried out by using a recurrence relation. The recurrence relation modifies the design variables so that in the design space the initial design is moved towards a design that satisfies the optimality criterion. The recurrence relation contains two sets of unknown terms. The first set is related to the gradient of the constraints and the second set is the Lagrange multipliers. It is necessary to determine these unknowns before the recurrence relation can be used.

The efficiency and the convergence behavior of the algorithm depend on (1) the recurrence relation used to modify the design variables, (2) the nature of the approximations made to derive the mathematical expression for the unknowns in the recurrence relations, and (3) how these unknowns are determined.

In deriving the optimality criterion and the algorithm based on it, we will assume that the structure is discretized into a number of elements, and that it is suitable for finite element analysis. In addition we will assume that the only design variables are the cross-sectional areas of the bar elements and the thicknesses of the plate elements. The geometry of the structure, materials, the loads applied to it are fixed.

This report discusses different methods developed to optimize structures with constraints on displacements, stresses, minimum sizes and general instability. The detailed discussion is primarily devoted to constraints on stresses and displacements, since these are the basic constraints in any structure. First, the equations of the displacement method of finite element analysis are reviewed; then the optimization methods are presented. Also discussed is the effect of the reciprocal

design variable on the optimality criterion and the optimization algorithms. The general optimization procedure and the different approaches that one can use to optimize a structure are then presented. The report concludes with illustrative problems indicating the effect of the different algorithms, then a summary, then an appendix. The appendix contains details of a three-bar truss problem solved by using different algorithms.

SECTION II

BASIC EQUATIONS OF ANALYSIS

DISPLACEMENT METHOD

The equations of finite element analysis which we will use to derive the optimality criterion and the optimization algorithm are summarized in this section.

The load vector $\{P\}$ is related to the displacement vector $\{u\}$ by the equation

$$[K]\{u\} = \{P\} \tag{1}$$

where $[K]$ is the global stiffness matrix of the structure. The strain energy V_i stored in the ith element is given by

$$V_i = \{u\}_i^t [k]_i \{u\}_i \tag{2}$$

where $\{u\}_i$ is the displacement vector associated with the ith element and $[k]_i$ is the ith element stiffness matrix. The ith element strain vector $\{\varepsilon\}_i$ is related to $\{u\}_i$ by

$$\{\varepsilon\}_i = [B]_i \{u\}_i \tag{3}$$

where $[B]_i$ is the strain-displacement matrix associated with the ith element.

The relation between the stress vector $\{\sigma\}_i$ and the strain vector $\{\varepsilon\}_i$ in the ith element is given by

$$\{\sigma\}_i = [E]_i \{\varepsilon\}_i \tag{4}$$

where $[E]_i$ is the stress-strain matrix of the elastic constants.

Using Eqs. 3 and 4, a relation between $\{\sigma\}_i$ and $\{u\}_i$ can be written as

$$\{\sigma\}_i = [D]_i \{u\}_i \tag{5}$$

where $[D]_i$ is the stress displacement matrix and is given by

$$[D]_i = [E]_i [B]_i \tag{6}$$

The displacement u_j at the jth node point in the structure is given by

$$u_j = \sum_{i=1}^{n} \frac{Q_{ij}}{A_i} \tag{7}$$

or

$$u_j = \sum_{i=1}^{n} Q_{ij} z_i \tag{8}$$

where n is the number of elements and z_i is the reciprocal design variable related to the direct design variable A_i by

$$z_i = \frac{1}{A_i} \tag{9}$$

In Eqs. 7 and 8, Q_{ij} is the flexibility coefficient and is given by

$$Q_{ij} = A_i \{u\}_i^t [k]_i \{s^j\}_i \tag{10}$$

or

$$Q_{ij} = \frac{1}{z_i} \{u\}_i^t [k]_i \{s^j\}_i \tag{11}$$

where $\{s^j\}_i$ is the virtual-displacement vector associated with the virtual-load vector $\{S^j\}$. The displacement u_j is also given by

$$u_j = \{u\}^t \{S^j\} \tag{12}$$

For a bar structure, the deflection at a point is given by the well-known relation

$$C_j = \sum_{i=1}^{n} \frac{T_i t_i^j \ell_i}{A_i E_i} \tag{13}$$

and the flexibility coefficient Q_{ij} can be written as

$$Q_{ij} = \frac{T_i t_i^j \ell_i}{E_i} \tag{14}$$

where T_i and t_i^j are the forces in the ith bar due to the applied load {P} and the virtual load $\{S^j\}$ respectively.

The linear stability of the structure is determined by solving the eigenvalue problem

$$[[K] + \mu[K_G]]\{\eta\} = 0 \tag{15}$$

where $[K_G]$ is the geometric stiffness matrix of the structure, and μ is the eigenvalue. The eigenvalue μ_j is given by

$$\mu_j = - \frac{\{\eta\}_j^t[K]\{\eta\}_j}{\{\eta\}\ [K_G]\{\eta\}_j} \tag{16}$$

where $\{\eta\}_j$ is the buckling mode. The buckling load of the structure is given by $\bar{\mu}\{P\}$, where $\bar{\mu}$ is the lowest eigenvalue.

SCALING OF THE DESIGN

In an optimization algorithm, it is convenient to obtain a feasible design after each iteration. This helps keep track of the reduction in the weight of the structure after each iteration and also helps to pick the most active constraints. A feasible design can be obtained by scaling the design to satisfy the specified constraints. The design variable A_i can be written as

$$A_i = \Lambda\alpha_i \tag{17}$$

where Λ is the scaling parameter. In Eq. 17, α_i is the relative value of the design variable A_i. The scaling parameter Λ is the same for all the elements. Using the definition of A_i given in Eq. 17, Eq. 1 can be written as

$$\Lambda[K']\{u\} = \{P\} \tag{18}$$

where [K'] is the stiffness matrix of the structure obtained by using the relative design vector $\{\alpha\}$. If $\{u'\}$ is the relative displacement vector, then the equilibrium equation can be written also as

$$[K']\{u'\} = \{P\} \tag{19}$$

Comparing Eqs. 18 and 19, the relationship between the actual displacement vector $\{u\}$ and the relative displacement vector $\{u'\}$ can be written as

$$\{u'\} = \Lambda\{u\} \tag{20}$$

Similarly, expressing the strain-displacement relation and the stress-strain relation in terms of the relative design variable $\{\alpha\}$, the relationship between the relative strains and stresses can be obtained. This gives

$$\{\varepsilon'\}_i = \Lambda\{\varepsilon\}_i \tag{21}$$

$$\{\sigma'\}_i = \Lambda\{\sigma\}_i \tag{22}$$

where $\{\varepsilon'\}_i$ and $\{\sigma'\}_i$ are the relative strains and stresses associated with the ith element. Eqs. 20 through 22 show that one can select the parameter Λ to satisfy the displacement constraint at either a node point or the stress constraint in an element. The Λ corresponding to the most critical constraint also will satisfy all other constraints. The design with the actual values of the design variables satisfying all constraints is given by Eq. 17, where Λ is determined for the most critical constraint. The simple scaling procedure discussed here can be used when the stiffness matrix is a linear function of the design variable. In other cases, one must iterate to obtain an acceptable scaling parameter. For certain problems, the scaling procedure may not be employable.

If the scaling parameter Λ is introduced in the stability equation (Eq. 15), we obtain

$$[[K'] + \mu'[K_G']]\{\eta'\} = 0 \tag{23}$$

where

$$\mu' = \frac{\mu}{\Lambda} \tag{24}$$

and

$$\{\eta'\} = \Lambda\{\eta\} \tag{25}$$

Using Eq. 24, we can scale the design so that the buckling load of the structure is equal to the applied load.

SECTION III

DISPLACEMENT CONSTRAINTS

The displacement constraints are fundamental to the development of an optimization algorithm that uses the displacement method of finite element analysis. The algorithms for other types of constraints are similar to that of the displacement-constrained problem. In the case of a stress-constrained problem, for example, an effective algorithm can be developed only by replacing the stress constraints by the equivalent displacement constraints. We will first consider the problem of designing a structure with a single-displacement constraint; then discuss the multiple constraints. The index 'i' in this section and subsequent sections refers to an element and goes from 1 to n where n is the number of elements.

SINGLE-DISPLACEMENT CONSTRAINT

Optimality Criterion

The weight of the structure can be written as

$$W(A) = \sum_{i=1}^{n} \rho_i A_i \ell_i \tag{26}$$

where ρ_i is the specific weight, A_i is the design variable, and ℓ_i is a function of the geometry of an element. The product $A_i \ell_i$ represents the volume of the element. For a bar element, A_i is the cross-sectional area and, for a plate element, it is the thickness. Eq. 26 assumes that the weight of the structure is a linear function of the design variable A_i.

A single-displacement constraint can be written as:

$$g_1 = C_1 - \overline{C}_1 = 0 \tag{27}$$

or

$$g_1 = \sum_{i=1}^{n} \frac{Q_{i1}}{A_i} - \overline{C}_1 = 0 \tag{28}$$

or

$$g_1 = \{u\}^t \{S^1\} - \overline{C}_1 = 0 \tag{29}$$

where $\overline{C}_1$ is the limiting value of the displacement.
Using Eqs. 26 and 29, the Lagrangian for a single-displacement constrained problem can be written as

$$L(A, \lambda) = \sum_{i=1}^{n} \rho_i A_i \ell_i + \lambda_1 \left(\sum_{i=1}^{n} \frac{Q_{i1}}{A_i} - \overline{C}_1 \right) \tag{30}$$

Differentiating this equation with respect to the design variable A_i and setting it equal to zero gives

$$\rho_i \ell_i - \lambda_1 \frac{Q_{i1}}{A_i^2} = 0 \tag{31}$$

This equation can be rewritten as

$$A_i^2 = \lambda_1 \frac{Q_{i1}}{\rho_i \ell_i} \tag{32}$$

or

$$1 = \lambda_1 \frac{Q_{i1}}{A_i^2 \rho_i \ell_i} \tag{33}$$

or

$$A_i = \lambda_1 \frac{Q_{i1}}{A_i \rho_i \ell_i} \tag{34}$$

or

$$1 = \lambda_1 \frac{e_{i1}}{\rho_i} \tag{35}$$

In Eq. 35, e_{i1} is the virtual strain energy density. Eqs. 31 through 35 represent the optimality criterion for a single-constrained problem. Eq. 35 states that in the optimum structure the ratio of virtual strain energy density to specific weight is the same for all the elements.

The flexibility coefficients Q_{i1} are constant for a statically determinate structure. In the case of a statically indeterminate structure,

these coefficients are functions of the design variables. In the derivation of the optimality criterion above, we seem to have treated Q_{i1} as a constant with respect to the design variable. However, even without this assumption an identical criterion can be derived. Consider the definition of the displacement constraint as given in Eq. 29. Using this equation and Eq. 1, we can write

$$C_1 = \{u\}^t[K]\{s^1\} \tag{36}$$

where $\{s^1\}$ is the virtual-displacement vector associated with the applied virtual-load vector $\{S^1\}$. Differentiating Eq. 36 with respect to A_i gives

$$\begin{aligned}\frac{\partial C_1}{\partial A_i} = \left\{\frac{\partial}{\partial A_i}\{u\}^t\right\}[K]\{s^1\} + \{u\}^t\left[\frac{\partial}{\partial A_i}[K]\right]\{s^1\} \\ + \{u\}^t[K]\left\{\frac{\partial}{\partial A_i}\{s^1\}\right\}\end{aligned} \tag{37}$$

Eq. 1 can be written as

$$\{u\}^t[K] = \{P\} \tag{38}$$

Differentiating this equation with respect to A_i gives

$$\left\{\frac{\partial}{\partial A_i}\{u\}^t\right\}[K] + \{u\}^t\left[\frac{\partial}{\partial A_i}[K]\right] = 0 \tag{39}$$

This equation can be written as

$$\frac{\partial}{\partial A_i}\{u\}^t = -\{u\}^t\left[\frac{\partial}{\partial A_i}[K]\right][K]^{-1} \tag{40}$$

Using this relation and a similar one for the gradient of the virtual-displacement vector $\{s^1\}^t$, Eq. 37 can be written as

$$\frac{\partial C_1}{\partial A_i} = - \{u\}^t \left[\frac{\partial}{\partial A_i} [K]\right] \{s^1\} + \{u\}^t \left[\frac{\partial}{\partial A_i} [K]\right] \{s^1\}$$

$$- \{u\}^t \left[\frac{\partial}{\partial A_i} [K]\right] \{s^1\} \tag{41}$$

$$= - \{u\}^t \left[\frac{\partial}{\partial A_i} [K]\right] \{s^1\}$$

Since $\frac{\partial}{\partial A_i} [K] = \frac{[k]_i}{A_i}$, Eq. 41 becomes

$$\frac{\partial C_1}{\partial A_i} = - \frac{\{u\}_i^t [k]_i \{s^1\}_i}{A_i} = - \frac{Q_{i1}}{A_i^2} \tag{42}$$

This derivation of the gradient of the constraint shows that in deriving the optimality criterion it is not necessary to assume Q_{i1} as a constant.

Recurrence Relations

The optimality criterion derived in the foregoing section is valid only at the optimum and has to be converted into a recurrence relation so that it can be used in an optimization algorithm. A recurrence relation can be written by multiplying both sides of Eq. 33 by A_i^r and taking the rth root. This gives

$$A_j^{k+1} = A_i^k \left(\lambda_1 \frac{Q_{i1}}{A_i^2 \rho_i \ell_i}\right)_k^{1/r} \tag{43}$$

where k+1 and k are introduced to indicate the iteration numbers. The parameter r in Eq. 43 determines the step size. We will call this an exponential recurrence relation. Eq. 43 can also be written as

$$A_i^{k+1} = A_i^k \left(1 + \left(\lambda_1 \frac{Q_{i1}}{A_i^2 \rho_i \ell_i} - 1 \right) \right)_k^{1/r} \tag{44}$$

In this equation $\lambda_1 \dfrac{Q_{i1}}{A_i^2 \rho_i \ell_i}$ is equal to unity at the optimum; thus,

near the optimum $\left(\lambda_1 \dfrac{Q_{i1}}{A_i^2 \rho_i \ell_i} - 1 \right)$ is small compared to unity. Therefore

using the binomial theorem to expand the right side of Eq. 44 and retaining only the linear terms gives

$$A_i^{k+1} = A_i^k \left(1 + \frac{1}{r} \left(\lambda_1 \frac{Q_{i1}}{A_i^2 \rho_i \ell_i} - 1 \right) \right)_k \tag{45}$$

This equation we will call a linear recurrence relation. In this equation $\left(\lambda_1 \dfrac{Q_{i1}}{A_i^2 \rho_i \ell_i} - 1 \right)$ is the error in satisfying the optimality criterion.

The exponential and linear recurrence relations can be expressed also in terms of the virtual strain energy density e_{i1} used to write the optimality criterion in Eq. 35. To use the recurrence relation, one must select the proper step-size parameter. For most problems, r=2 is generally adequate. Increasing the value of r reduces the step size.

The recurrence relations contain two unknowns: the flexibility coefficients Q_{i1} and the Lagrange multiplier λ_1. The coefficients Q_{i1} can be determined by using Eq. 10. The Lagrange multiplier λ_1 can be obtained by using the relations derived in the next section.

Expressions for the Lagrange Multiplier

The design variable A_i using Eq. 32 can be written as

$$A_i = \sqrt{\lambda_1 \frac{Q_{i1}}{\rho_i \ell_i}} \tag{46}$$

The Lagrange multiplier λ_1 can be evaluated by substituting Eq. 46 in Eq. 28 and solving for λ_1. This gives

$$\sqrt{\lambda_1} = \frac{\sum_{i=1}^{n} \sqrt{Q_{i1} \rho_i \ell_i}}{\overline{C}_1} \tag{47}$$

We can derive two more expressions for the Lagrange multiplier by using the optimality criterion and the constraint equations. Substituting Eq. 34 in Eq. 26 and using Eq. 28, the weight of the structure is given by

$$W = \lambda_1 \overline{C}_1 \tag{48}$$

or

$$\lambda_1 = \frac{W}{\overline{C}_1} \tag{49}$$

Eq. 48 states that at the optimum the weight of the structure is equal to the product of the Lagrange multiplier λ_1 and the limiting value of the constrained $\overline{C}_1$.

Another expression for the Lagrange multiplier can be obtained by writing the optimality criterion as

$$\frac{1}{A_i} = \lambda_1 \frac{Q_{i1}}{\rho_i \ell_i A_i^3} \tag{50}$$

and substituting it in Eq. 28. This gives

$$\lambda_1 = \frac{\overline{C}_1}{\sum_{i=1}^{n} \frac{Q_{i1}^2}{\rho_i \ell_i A_i^3}} \quad (51)$$

Eliminating λ_1 from Eqs. 48 and 51, the weight of the structure at the optimum is given by

$$W = \frac{\overline{C}_1^2}{\sum_{i=1}^{n} \frac{Q_{i1}^2}{\rho_i \ell_i A_i^3}} \quad (52)$$

The Lagrange multiplier λ_1 in Eqs. 43 and 45 can be eliminated by using one of the expressions for λ_1 derived above. However, in the case of the single-constrained problem if the structure can be scaled by using Eqs. 17 through 22, a relative value of the Lagrange multiplier can be used. Therefore, one can set λ_1 equal to unity. The expressions for the Lagrange multiplier derived in this section are strictly valid only at the optimum.

Effect of the Passive Elements on the Lagrange Multiplier

In deriving the expressions for the Lagrange multipliers for the single-displacement constraint we have assumed that all the design variables can be modified by using the recurrence relation. In a practical design problem there are always constraints on the minimum and maximum sizes of the elements. These elements, or other elements whose sizes are not modified during the iterations by using the recurrence relation are called passive elements. One must modify the algorithm to account for these elements. When the exponential recurrence relation is used, if the virtual strain energy associated with any element is negative, then that element is sized by some other criterion and becomes a passive element. Elements whose sizes are governed by the recurrence

relation can be called active elements. At the optimum the optimality criterion is satisfied only by the active elements.

The contribution to the weight of the structure and the constraint can be divided into two parts, one due to the active and the other due to the passive elements. Thus the weight of the structure and the constraint equation can be written as

$$W = \sum_{i=1}^{n_1} \rho_i A_i \ell_i + W^* \tag{53}$$

and

$$g_1 = \sum_{i=1}^{n_1} \frac{Q_{i1}}{A_i} + C_1^* - \bar{C}_1 = 0 \tag{54}$$

where W^* is the weight of the passive elements and C_1^* is the contribution of the passive elements to the constraint. In Eqs. 53 and 54, n_1 is the number of active elements. This change in the definition of the weight of the structure and the constraint affects the expression for the Lagrange multiplier. With this change Eqs. 47, 49 and 51 can be written, respectively, as

$$\sqrt{\lambda_1} = \frac{\sum_{i=1}^{n_1} \sqrt{Q_{i1}\rho_i \ell_i}}{\bar{C}_1 - C_1^*} \tag{55}$$

$$\lambda_1 = \frac{W - W^*}{\bar{C}_1 - C_1^*} \tag{56}$$

$$\lambda_1 = \frac{\bar{C}_1 - C_1^*}{\sum_{i=1}^{n_1} \frac{Q_{i1}^2}{\rho_i \ell_i A_i^3}} \tag{57}$$

Hence, when there are passive elements we should use Eqs. 55, 56, and 57 instead of 47, 49, and 51, respectively.

In the case of a single-displacement constrained problem the explicit expression for the Lagrange multiplier does not have a significant effect on the algorithm. However, in the multiple-displacement constrained problem, the method used to determine the Lagrange multipliers plays a major role in the optimization algorithm.

MULTIPLE-DISPLACEMENT CONSTRAINTS

The method discussed in the last section to design a minimum weight structure would give a minimum weight design only if (1) a structure is subjected to a single displacement constraint, or (2) only one constraint is active at the optimum. The active constraints may be defined as those that are satisfied as equality constraints at the optimum. The multiple-constrained problem is much more difficult to solve than the single-constrained problem. The main reason for this is that a simple explicit expression cannot be derived for the Lagrange multipliers. Also one must develop the logic to predict a probable set of active constraints during the iterations. If all the active constraints are not taken into consideration, the convergence behavior is affected. If all the constraints imposed on the structure are considered potentially active, then the algorithm becomes inefficient from the point of view of computational effort. This is particularly true when there are a large number of inequality constraints imposed on the structure.

If the structure is subjected to more than one loading condition, the loading conditions can be treated as multiple constraints. So to simplify the notation we will not consider the multiple-loading conditions in defining the problem.

The Lagrangian for the multiple-displacement constrained problem can be written as

$$L(A, \lambda) = \sum_{i=1}^{n} \rho_i A_i \ell_i + \sum_{j=1}^{m} \lambda_j g_j \tag{58}$$

where

$$g_j = C_j - \bar{C}_j \leq 0 \qquad j=1, m \tag{59}$$

and

$$C_j = \sum_{i=1}^{n} \frac{Q_{ij}}{A_i} \tag{60}$$

or

$$C_j = \{u\}^t\{S^j\} \tag{61}$$

In Eqs. 58 and 59, g_j are the inequality constraints, λ_j are the Lagrange multipliers, and m is the number of inequality constraints.

The optimality criterion can be derived by differentiating the Lagrangian with respect to the design variables A_i and setting the resulting equations to zero. This gives

$$\rho_i \ell_i - \sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{A_i^2} = 0 \tag{62}$$

where $\lambda_j g_j = 0$. For the active constraints, $\lambda_j > 0$ and $g_j = 0$ and, for the passive constraints, $\lambda_j = 0$ and $g_j \neq 0$. The optimality criterion in Eq. 62 can be written in different forms as follows:

$$A_i^2 = \sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i} \tag{63}$$

or

$$A_i = \sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i} \tag{64}$$

or

$$1 = \sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} \tag{65}$$

or

$$1 = \sum_{j=1}^{m} \lambda_j \frac{e_{ij}}{\rho_i} \tag{66}$$

or

$$1 = \frac{\{u\}_i^t [k]_i \sum_{j=1}^{m} \lambda_j \{s^j\}_i}{\rho_i \ell_i A_i} \tag{67}$$

or

$$1 = \frac{\bar{e}_{ij}}{\rho_i} \tag{68}$$

where

$$\bar{e}_{ij} = \frac{\{u\}_i^t [k]_i \sum_{j=1}^{m} \lambda_j \{s^j\}_i}{A_i \ell_i} \tag{69}$$

The optimality criterion as written in Eqs. 66 and 68 has a physical interpretation. According to Eq. 66, at the optimum the weighted sum of the ratio of virtual strain energy density to mass density is equal to unity for all the elements, where the weighting parameters are the Lagrange multipliers. The virtual strain energy density in each element is due to a virtual load vector associated with each distinct constraint. Eq. 68 which is also an optimality criterion states that the virtual strain energy to mass density is equal to unity for all elements where a single virtual load vector equal to $\sum_{j=1}^{m} \lambda_j \{S^j\}$ is applied to the structure. In using the recurrence relation based on Eq. 65 or 66, one must determine in each element the virtual strain energy due to m virtual loads. But if the recurrence relation based on Eq. 67 is used, it is only necessary to consider one virtual load vector to determine the virtual strain energy in an element. However, one must know the actual or relative values of the Lagrange multipliers before a single virtual load vector can be assembled. This is not always possible as we will see when we discuss the methods to determine the Lagrange multipliers.

The 'n' optimality conditions and the 'm' constraints must be satisfied by the optimum design. These are (m+n) nonlinear equations that must be solved to determine the 'n' design variables and the 'm'

Lagrange multipliers. The optimization algorithm solves these nonlinear equations by an iterative scheme. In an iterative method the design variables are modified by using the recurrence relation so that the optimality criterion is satisfied. The Lagrange multipliers are evaluated on the basis that the constraint conditions are satisfied when the design variables are changed. Since the Lagrange multipliers and the design variables are functions of each other, any change in one nonlinearly affects the other, and this necessitates the use of an iterative method.

Recurrence Relations for Multiple Constraints

The recurrence relations for the multiple displacement constrained problem can also be divided into two categories: (1) an exponential form, and (2) a linear form, as we have done for the single-displacement constrained problem. The method of deriving them is similar to the one we used to write Eqs. 43 and 45 for the single-constrained problem.

An exponential recurrence relation can be written by multiplying Eq. 65 by A_i^r and taking the rth root. This gives

$$A_i^{k+1} = A_i^k \left(\sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} \right)_k^{1/r} \tag{70}$$

where k+1 and k are the iteration numbers. In Eq. 70 the parameter r determines the step size. Using the same argument given to derive Eq. 45, we can write a linear recurrence relation for the multiple-displacement-constrained problem. This gives

$$A_i^{k+1} = A_i^k \left(1 + \frac{1}{r} \left(\sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} - 1 \right) \right)_k \tag{71}$$

This equation can also be written as

$$A_i^{k+1} = A_i^k + \frac{1}{r} \left(\sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i} - A_i \right)_k \tag{72}$$

In this equation the term in parenthesis is the correction needed to the design variable A_i to satisfy the optimality criterion as defined in Eq. 64. Thus we can say that the objective of modifying the design variable by using the recurrence relation is to move the current design towards a design that satisfies the optimality criterion. When the step-size parameter r=1, Eqs. 70 and 71 reduce to the same equation. For the multiple-constrained problem, as in the case of a single constraint, r=2 can be considered as the normal step size, but some problems require one to increase r to reduce the step size. In the exponential recurrence relation the design variable is modified by multiplying it by a quantity which is equal to unity at the optimum, and in the linear recurrence relation the design variable is modified by adding a quantity which is equal to zero at the optimum. Eqs. 70 and 71 can also be written by using the optimality criterion defined by Eqs. 66 and 68.

Methods to Determine the Lagrange Multipliers

In order to use any one of the recurrence relations the coefficients Q_{ij} and the Lagrange multipliers have to be known. The coefficients Q_{ij} can be determined by using Eq. 10. The equations to determine the Lagrange multipliers can be derived by considering the effect of a change in the design variables or Lagrange multipliers on the constraint equations. We will derive in this section different methods, some rigorous and some not rigorous. The rigorous methods give equations that are reliable but the methods need more computational effort. The nonrigorous methods are simple and need less computational effort, butgive algorithms which may not lead to a minimum or are slow to converge. In discussing the different methods we will point out their advantages and disadvantages and their relationship to one another.

Recurrence Relations for the Lagrange Multipliers

A recurrence relation to estimate the Lagrange multipliers can be written by assuming that all the constraints in Eq. 59 are equality constraints. This gives

$$C_j = \overline{C}_j \tag{73}$$

Multiplying both sides of this relation by λ_j^b and taking the bth root, a recurrence relation can be written as

$$\lambda_j^{k+1} = \lambda_j^k \left(\frac{C_j}{\overline{C}_j} \right)^{1/b} \tag{74}$$

where k refers to the iteration number and the parameter b controls the step size. The advantages of using this recurrence relation are:

(1) An equivalent single virtual load vector can be used to determine the virtual strain energy in an element, and individual values of Q_{ij} for different constraints need not be determined.

(2) It is not necessary to predict which constraints are potentially active. Repeated use of Eq. 74 reduces the value of a Lagrange multiplier corresponding to a passive constraint, and its contribution to the virtual load vector is reduced after each iteration.

(3) The computational effort required to determine the Lagrange multipliers and the virtual strain energy in an element is minimal as compared to other methods.

The disadvantages are (1) Initial values of the Lagrange multipliers have to be assumed, and (2) Convergence to the minimum-weight design is slow and large oscillations sometimes occur in the scaled weight of the structure.

The recurrence relation in Eq. 74 can be written as

$$\lambda_j^{k+1} = \lambda_j^k \left(1 + \left(\frac{\overline{C}_j}{C_j} - 1 \right) \right)_k^{-1/b} \tag{75}$$

Since at the optimum $\frac{\overline{C}_j}{C_j}$ is nearly equal to unity, the difference $\left(\frac{\overline{C}_j}{C_j} - 1 \right)$ is small as compared to unity. We can therefore expand Eq. 75 by using the Binomial theorem and retain only the linear terms. This gives

$$\lambda_j^{k+1} = \lambda_j^k \left(\frac{b+1}{b} - \frac{1}{b} \frac{\overline{C}_j}{C_j} \right)_k \tag{76}$$

This equation can also be written as

$$\bar{C}_j - C_j^k = bC_j^k \left(1 - \frac{\lambda_j^{k+1}}{\lambda_j^k} \right) \tag{77}$$

We will show in the next section that this linear recurrence relation for the Lagrange multipliers is an approximation to a set of linear equations that can be used to determine the Lagrange multipliers.

Linear Equations to Determine the Lagrange Multipliers

A set of equations to determine the Lagrange multipliers can be obtained by considering the change in the constraint due to a change in the design variable A_j. The change in the jth constraint can be written as

$$\Delta g_j = g_j(A + \Delta A) - g_j(A) \tag{78}$$

$$= \sum_{i=1}^{n} \frac{\partial g_j}{\partial A_i} \Delta A_i \tag{79}$$

If the change in the design variable is assumed to be such that the constraints at point {A+dA} are satisfied i.e. $g_j(A+dA)=0$, then using Eq. 59, Eq. 79 can be written as

$$\bar{C}_j - C_j^k = \sum_{i=1}^{n} \frac{\partial g_j}{\partial A_i} \Delta A_i^k \tag{80}$$

where k refers to the iteration number. Differentiating Eq. 60 with respect to the design variable A_i and assuming that Q_{ij} is a constant gives

$$\frac{\partial g_j}{\partial A_i} = - \frac{Q_{ij}}{A_i^2} \tag{81}$$

The change in the design variable from the kth to the (k+1)th iteration is given by

$$\Delta A_i^k = A_i^{k+1} - A_i^k \tag{82}$$

Substituting A_i^{k+1} from Eq. 71 in this equation gives

$$\Delta A_i^k = \frac{1}{r}\left(\sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} - 1 \right)_k A_i^k \tag{83}$$

Using Eqs. 81 and 83, Eq. 80 can be written as

$$\sum_{p=1}^{m} \lambda_p^{k+1} \left(\sum_{i=1}^{n} \frac{Q_{ij} Q_{ip}}{\rho_i \ell_i A_i^3} \right)_k = r\,(C_j^k - \overline{C}_j) + \sum_{j=1}^{n} \left(\frac{Q_{ij}}{A_i} \right)_k \tag{84}$$

Since $\sum_{i=1}^{n} \left(\frac{Q_{ij}}{A_i} \right)_k = C_j^k$, Eq. 84 can be written as

$$\sum_{p=1}^{m} \lambda_p^{k+1} \left(\sum_{i=1}^{n} \frac{Q_{ij} Q_{ip}}{\rho_i \ell_i A_i^3} \right) = (r+1) C_j^k - r\overline{C}_j \tag{85}$$

These are 'm' equations corresponding to the 'm' displacement constraints, which can be used to determine the Lagrange multipliers in the iterative algorithm. At each iteration one must use only those equations corresponding to the active constraints, giving positive Lagrange multipliers. This can generally be achieved by considering only those constraints that are closest to the constraint surface and by eliminating the equations yielding negative Lagrange multipliers.

When there are passive elements whose sizes are not governed by the optimality criterion, we have to separate the contribution of those elements in the summation of Eq. 84. This gives

$$\sum_{p=1}^{m} \lambda_p^{k+1} \left(\sum_{i=1}^{n_1} \frac{Q_{ij} Q_{ip}}{\rho_i \ell_i A_i^3} \right)_k = r(C_j^k - \overline{C}_j) + \sum_{i=1}^{n_1} \left(\frac{Q_{ij}}{A_i} \right)_k - r \sum_{i=n_1+1}^{n} \left(\frac{Q_{ij}}{A_i^2} \right) \Delta A_i^k(p) \tag{86}$$

where 'n_1' is the number of active elements and $\Delta A_i^k(p) = A_i^k(p) - A_i^k$. $A_i^k(p)$ is the size of a passive element dictated by considerations other than the optimality criterion. Generally the passive elements are those whose sizes are governed by a minimum or a maximum size. For a problem with stress and displacement constraints, if we decide to treat the stress constraints as passive constraints, then the elements whose sizes are governed by stresses will be included in the passive category.

The advantages of using Eqs. 85 and 86 to determine the Lagrange multipliers are:

(1) It is not required to assume initial values of the Lagrange multipliers, since the Lagrange multipliers are evaluated by solving the equations.

(2) Since the equations contain coupling terms which take into account the interdependence of the different constraints, the Lagrange multipliers are more realistic. This is particularly true in the case of a structure where the constraints are sensitive to design changes.

(3) The convergence behavior is generally better than other approaches.

The disadvantages are:

(1) A single virtual load vector cannot be assembled since the Lagrange multipliers are not known before the flexibility coefficients Q_{ij} corresponding to the constraints are determined.

(2) The computational effort, to determine the flexibility coefficients and to assemble the coefficients of the Lagrange multipliers in Eqs. 84, 85, and 86 substantially increases with an increase in the number of potentially active constraints.

(3) It is necessary to solve a set of simultaneous equations.

(4) Some scheme has to be used in order to determine the potentially active constraints and to eliminate those equations corresponding to the passive constraints.

At the optimum the active constraints are satisfied as equality constraints, i.e. $C_j=\overline{C}_j$ and Eq. 85 becomes

$$\sum_{p=1}^{m} \lambda_p^{k+1}\left(\sum_{i=1}^{n} \frac{Q_{ij}Q_{ip}}{\rho_i \ell_i A_i^3}\right) = \overline{C}_j \tag{87}$$

This relation, although valid only at the optimum, can be used to estimate the Lagrange multipliers. Eq. 87 can also be derived directly by using the optimality criterion and the constraint equations. The optimality criterion can be written as

$$\{I\}_n = [Q]_{nxm} \{\lambda\}_m \tag{88}$$

The constraint equations can be written as

$$[F]_{mxn} \{I\}_n = \{\overline{C}\}_m \tag{89}$$

In Eqs. 88 and 89, $\{I\}$ is a vector with all the elements equal to unity. Using these equations we can write

$$[F]_{mxn} [Q]_{nxm} \{\lambda\}_m = \{\overline{C}\}_m \tag{90}$$

or

$$[R]_{mxm} \{\lambda\}_m = \{\overline{C}\}_m \tag{91}$$

where $[R]_{mxm} = [F]_{mxn} [Q]_{nxm}$

In Eq. 91 the elements of [R] are given by

$$R_{jp} = \sum_{i=1}^{n} \frac{Q_{ij}Q_{ip}}{\rho_i \ell_i A_i^3} \tag{92}$$

Comparing Eqs. 87 and 91 it is seen that they are identical.

Eq. 87 can also be derived by writing the optimality criterion as

$$\frac{1}{A_i} = \sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^3} \tag{93}$$

and substituting it into the constraint equation

$$\sum_{i=1}^{n} \frac{Q_{ij}}{A_i} = \bar{C}_j \tag{94}$$

The recurrence relation for the Lagrange multipliers given by Eq. 76 can be shown to be an approximation to the linear equations in Eq. 85. The matrix multiplying the vector $\{\lambda\}$ in Eq. 85 is square. If we neglect the off-diagonal terms, these equations become

$$\lambda_j^{k+1} \sum_{i=1}^{n} \left(\frac{Q_{ij} Q_{ij}}{p_i \ell_i A_i^3} \right)_k = (r+1) C_j^k - r\bar{C}_j \tag{95}$$

These equations are uncoupled and assume that the constraints are independent of one another. With this assumption, and using the optimality criterion, $(Q_{ij}/p_i \ell_i A_i^3)_k$ in Eq. 95 can be replaced by $1/\lambda_j^k$. Also recalling that $\left(\sum_{i=1}^{n} \frac{Q_{ij}}{A_i} \right)_k = C_k^k$, Eq. 95 can be written as

$$\frac{\lambda_j^{k+1}}{\lambda_j^k} C_j^k = (r+1) C_j^k - r\bar{C}_j \tag{96}$$

or

$$\lambda_j^{k+1} = \lambda_j^k \left((r+1) - r \frac{\bar{C}_j}{C_j} \right)_k \tag{97}$$

This equation would be identical with Eq. 76 if

$$\frac{1}{b} = r \tag{98}$$

This shows that Eq. 76, which is a linearized form of Eq. 74 is an approximation to Eq. 85.

Newton-Raphson Method

An iterative algorithm to solve the optimality criterion and the constraint equations can be developed by considering the change in

a constraint due to a change in the Lagrange multipliers. The change in a constraint can be written as

$$\Delta g_j = g_j\,(\lambda + \Delta\lambda) - g_j(\lambda) = \sum_{j=1}^{m} \frac{\partial g_j}{\partial \lambda} \Delta\lambda \tag{99}$$

Since in the Newton Raphson method the change $\Delta\lambda$ is selected to satisfy the condition $g_j(\lambda+\Delta\lambda)=0$, using Eq. 99 an iterative relation can be written as

$$-\ \phi\{g\}_k = [H]\{\lambda^{k+1} - \lambda^k\} \tag{100}$$

or

$$\{\lambda\}^{k+1} = \{\lambda\}^k - \phi[H]_k^{-1}\{g\}_k \tag{101}$$

where $[H]^{-1}$ is the inverse of the Hessian [H] whose elements are

$$H_{jp} = \frac{\partial g_j}{\partial \lambda_p} \tag{102}$$

and ϕ is a parameter introduced to control the step size.

Differentiating Eq. 60 with respect to λ_j, recalling that λ_j is related to A_i by Eq. 63 and assuming Q_{ij} as constants, we can write

$$\frac{\partial g_j}{\partial \lambda_p} = -\frac{1}{2} \sum_{i=1}^{n} \frac{Q_{ij}Q_{ip}}{\rho_i \ell_i A_i^3} \tag{103}$$

Eq. 101 can be used to update the initially assumed Lagrange multipliers. The summation in Eq. 103 is carried out only over the active elements. The iterative process in this method consists of using Eq. 101 to estimate the Lagrange multipliers and Eq. 63 to modify the design variables alternatively until the constraint equations are satisfied within a specified tolerance. The disadvantages of this method are: (1) It is essential to assume initial values of the Lagrange multipliers, (2) The passive constraints cannot be easily separated from the active constraints,

and (3) If too many iterations are performed to satisfy the constraints, the design might move away from the region where the coefficient Q_{ij} are valid.

The Newton-Raphson algorithm, can be shown to be related to the linear equations in Eq. 85. Eq. 101 can be written as

$$[H]_k\{\lambda\}^{k+1} = [H]_k\{\lambda\}^k - \phi\{g\}_k \tag{104}$$

Using Eq. 103, this equation can be written as

$$\sum_{p=1}^{m} \lambda_p^{k+1} \sum_{i=1}^{n} \left(\frac{Q_{ij}Q_{ip}}{\rho_i \ell_i A_i^3}\right)_k = \sum_{p=1}^{m} \lambda_p^{k} \sum_{i=1}^{n} \left(\frac{Q_{ij}Q_{ip}}{\rho_i \ell_i A_i^3}\right)_k + 2\phi g_j^k \tag{105}$$

Substituting A_i^2 from Eq. 63 into Eq. 60 and rearranging, one obtains

$$g_j^k = \sum_{p=1}^{m} \lambda_p^{k} \sum_{i=1}^{n} \left(\frac{Q_{ij}Q_{ip}}{\rho_i \ell_i A_i^3}\right)_k - \bar{C}_j \tag{106}$$

Using this equation and recalling that $g_j^k = (C_j^k - \bar{C}_j)$, Eq. 105 can be written as

$$\sum_{p=1}^{m} \lambda_p^{k+1} \sum_{i=1}^{n} \left(\frac{Q_{ij}Q_{ip}}{\rho_i \ell_i A_i^3}\right)_k = (2\phi+1)C_j^k - 2\phi\bar{C}_j \tag{107}$$

This equation is identical to Eq. 85, for $2\phi=r$.

The advantage of using Eq. 107 in the Newton-Raphson method instead of Eq. 101 is that one need not assume the initial values of the Lagrange multipliers. Eq. 107 can be solved at each iteration. This approach we refer to as the modified Newton-Raphson method.

SECTION IV

STRESS CONSTRAINTS

An algorithm can be developed to design a structure with stress constraints by using two approaches. The first approach is to specify the constraint as a function of the stress in an element, and the second is to convert the stress constraint into an equivalent displacement constraint. These two approaches, depending on the approximations used to derive the criterion, lead to different algorithms. For convenience we will consider only bar elements, but the conclusions and the algorithms can be easily extended to other membrane-type elements.

FULLY STRESSED DESIGN (FSD) METHOD

The stress constraint in the ith element can be written as

$$g_i = \left(\frac{\sigma_i}{\bar{\sigma}_i} - 1 \right) \leq 0 \tag{108}$$

where σ_i is the actual stress and $\bar{\sigma}_i$ is the maximum allowable stress in the ith element. The number of constraints is equal to the number of elements. The stress in the ith element is given by

$$\sigma_i = \frac{T_i}{A_i} \tag{109}$$

where T_i is the force in the ith element. Using Eqs. 26, 108, and 109, the Lagrangian for the stress-constrained problem can be written as

$$L(A, \lambda) = \sum_{i=1}^{n} \rho_i A_i \ell_i + \sum_{i=1}^{n} \lambda_i \left(\frac{T_i}{A_i \bar{\sigma}_i} - 1 \right) \tag{110}$$

Differentiating this equation with respect to the design variable A_i and setting it equal to zero gives

$$\rho_i \ell_i - \lambda_i \frac{T_i}{A_i^2 \bar{\sigma}_i} + \frac{1}{A_i \bar{\sigma}_i} \sum_{p=1}^{n} \lambda_p \frac{\partial T_p}{\partial A_i} = 0 \tag{111}$$

In this equation the term $\frac{\partial T_p}{\partial A_i}$, which is the gradient of the force in a bar, cannot be explicitly written. For a determinate structure, this gradient is zero, since the force in a bar is independent of the areas of the areas of the elements. If we assume that $\sum_{p=1}^{n} \lambda_p \frac{\partial T_p}{\partial A_i} = 0$, which is not true for an indeterminate structure, Eq. 111 becomes

$$\rho_i \ell_i - \lambda_i \frac{T_i}{A_i^2 \bar{\sigma}_i} = 0 \tag{112}$$

or

$$A_i = \sqrt{\lambda_i} \sqrt{\frac{T_i}{\bar{\sigma}_i \rho_i \ell_i}} \tag{113}$$

Now if the stress constraints imposed on the structure are assumed to be satisfied as equality constraints, then using Eqs. 108 and 109 we can write

$$\frac{T_i}{A_i} = \bar{\sigma}_i \tag{114}$$

Substituting Eq. 113, in this equation and solving for λ_i gives

$$\sqrt{\lambda_i} = \sqrt{\frac{T_i \rho_i \ell_i}{\bar{\sigma}_i}} \tag{115}$$

Substituting this relation in Eq. 113 gives

$$A_i = \frac{T_i}{\bar{\sigma}_i} \tag{116}$$

This equation states that at the optimum the area of the element is equal to the force in the bar divided by the maximum allowable stress in the bar, or the stress in an element is equal to the maximum allowable stress in that element. This is the well-known Fully Stressed Design

optimality criterion. A recurrence relation based on this criterion can be written as

$$A_i^{k+1} = \left(\frac{T_i}{\bar{\sigma}_i} \right)_k \tag{117}$$

At the optimum $T_i = A_i\bar{\sigma}_i$. Substituting this in Eq. 113 we can write

$$\begin{aligned} \lambda_i &= A_i \rho_i \ell_i \\ &= \text{weight of the element} \end{aligned} \tag{118}$$

The use of the Fully Stressed Design algorithm, if applied to an indeterminate structure, is an approximation, since for this structure $\sum_{p=1}^{n} \lambda_p \frac{\partial T_p}{\partial A_i} \neq 0$. Therefore the FSD algorithm does not necessarily give a minimum weight design for an indeterminate structure, and this is particularly true when the allowable stress in all the elements is not the same. The design obtained by the Fully Stressed Design algorithm for an indeterminate structure with unequal stresses sometimes not only gives a non-optimum design but also gives a design with a bad distribution of material and an inefficient load path.

STRESS CONSTRAINT WITH EQUIVALENT DISPLACEMENT CONSTRAINT

The stress in the ith bar element can be written as

$$\sigma_i = [D_{11} \;\; D_{12}]_i \begin{bmatrix} U^1 \\ U^2 \end{bmatrix}_i \tag{119}$$

where U^1 and U^2 are the longitudinal displacement of the two nodes defining the ith bar. In Eq. 119, $D_{11} = -\frac{E_i}{\ell_i}$ and $D_{12} = \frac{E_i}{\ell_i}$ where E_i is the Young's modulus and ℓ_i is the length of the bar. The stress constraint in the ith element can now be written as

$$g_i = \sigma_i - \bar{\sigma}_i \leq 0 \tag{120}$$

or

$$= (D_{11}U^1 + D_{12}U^2)_i - \bar{\sigma}_i \leq 0 \tag{121}$$

Thus we have replaced the stress constraint in an element by an equivalent constraint on the linear combination of the displacements in the longitudinal direction at the two nodes defining the element. The constraint equation can now be written

$$g_j = \sigma_j - \bar{\sigma}_j \leq 0 \qquad j=1,\ldots n \tag{122}$$

or

$$g_j = \sum_{i=1}^{n} \frac{R_{ij}}{A_i} - \sigma_j \leq 0 \tag{123}$$

or

$$g_j = \{u\}^t\{S^j\} - \bar{\sigma}_j \leq 0 \tag{124}$$

where

$$R_{ij} = A_i\{u\}_i^t[k]_i\{s^j\}_i \tag{125}$$

There will be n constraints equal to the number of elements in a bar structure. In Eqs. 125 $\{s^j\}_i$ is the virtual-displacement vector associated with the ith element due to the virtual load vector $\{S^j\}$, which is equivalent to the forces D_{11} $(= - E_j/\ell_j)$ and D_{12} $(=E_j/\ell_j)$ acting in the longitudinal direction at the two nodes at the ends of jth element. Using Eqs. 26 and 123 the Lagrangian for the stress-constrained problem can be written as

$$L(A, \lambda) = \sum_{i=1}^{n} \rho_i A_i \ell_i + \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{n} \frac{R_{ij}}{A_i} - \bar{\sigma}_j \right) \tag{126}$$

Differentiating this relation with respect to the design variable and setting the result equal to zero gives

$$\rho_i \ell_i - \sum_{j=1}^{n} \lambda_j \frac{R_{ij}}{A_i^2} = 0 \tag{127}$$

or

$$1 = \sum_{j=1}^{n} \lambda_j \frac{R_{ij}}{\rho_i \ell_i A_i^2} \tag{128}$$

where

$$\lambda_j \geq 0 \text{ and } \lambda_j \bar{\sigma}_j = 0 \tag{129}$$

This optimality criterion derived for the stress constraints has the same form as the optimality criterion in Eq. 65 for the multiple-displacement-constrained problem. Hence following the same procedure and logic we can derive the recurrence relations and the equations to determine the Lagrange multipliers. The recurrence relations for the stress-constrained problem equivalent to Eqs. 70 and 71 are

$$A_i^{k+1} = A_i^k \left(\sum_{j=1}^{n} \lambda_j \frac{R_{ij}}{\rho_i \ell_i A_i^2} \right)_k^{1/r} \tag{130}$$

and

$$A_i^{k+1} = A_i^k \left(1 + \frac{1}{r} \left(\sum_{j=1}^{n} \lambda_j \frac{R_{ij}}{\rho_i \ell_i A_i^2} - 1 \right) \right)_k \tag{131}$$

Similarly the equations to determine the Lagrange multipliers equivalent to Eqs. 74, 76, 77, 85, and 87, respectively, are

$$\lambda_j^{k+1} = \lambda_j^k \left(\frac{\sigma_j}{\bar{\sigma}_j} \right)_k^{1/b} \tag{132}$$

$$\lambda_j^{k+1} = \lambda_j^k \left(\frac{b+1}{b} - \frac{1}{b} \frac{\bar{\sigma}_j}{\sigma_j} \right)_k \tag{133}$$

$$\bar{\sigma}_j - \sigma_j^k = b\sigma_j^k \left(1 - \frac{\lambda_j^{k+1}}{\lambda_j^k} \right)_k \tag{134}$$

$$\sum_{p=1}^{n} \lambda_p^{k+1} \sum_{i=1}^{n} \left(\frac{R_{ij} R_{ip}}{\rho_i \ell_i A_i^3} \right)_k = (r+1)\sigma_j^k - r\bar{\sigma}_j \tag{135}$$

$$\sum_{p=1}^{n_1} \lambda_p^{k+1} \sum_{i=1}^{n_1} \left(\frac{R_{ij}R_{ip}}{\rho_i \ell_i A_i^3}\right)_k = r(\sigma_j^k - \bar{\sigma}_j)$$

$$+ \sum_{i=1}^{n_1} \left(\frac{R_{ij}}{A_i}\right)_k - r \sum_{i=n_1+1}^{n} \left(\frac{R_{ij}}{A_i^2}\right) \Delta A_i^k(p) \tag{136}$$

$$\sum_{p=1}^{n} \lambda_p^{k+1} \sum_{i=1}^{n} \left(\frac{R_{ij}R_{ip}}{\rho_i \ell_i A_i^3}\right)_k = \bar{\sigma}_j \tag{137}$$

In the case of a stress-constrained problem, the number of constraints is equal to the total number of stress constraints imposed on the structure. For a bar structure the number of constraints will be equal to the number of elements. If the structure is idealized with membrane elements and stress constraints are imposed on σ_x, σ_y, σ_{xy}, there will be three constraints associated with each element.

In Eq. 137, if the off-diagonal terms multiplying the vector of Lagrange multipliers are neglected, this equation can be approximated as

$$\lambda_p^{k+1} \sum_{i=1}^{n} \left(\frac{R_{ip}R_{ip}}{\rho_i \ell_i A_i^3}\right)_k = \bar{\sigma}_p \; , \tag{138}$$

In addition if we assume that the virtual strain energy in an element is only due to the virtual load in that element, which is true for a statically determinate structure, then Eq. 138 becomes

$$\lambda_i^{k+1} \left(\frac{R_{ii}R_{ii}}{\rho_i \ell_i A_i^3}\right)_k = \bar{\sigma}_i \tag{139}$$

Since in this equation we are assuming that the elements are not interdependent, $R_{ii} = \dfrac{T_i \ell_i}{E_i}$. In addition if we assume that the elements

are fully stressed, i.e. $A_i = \frac{T_i}{\bar{\sigma}_i}$, a simple expression for the Lagrange multiplier can be written as

$$\lambda_i^{k+1} = \frac{T_i}{\bar{\sigma}_i^2}\left(\frac{\rho_i E_i^2}{\ell_i}\right) \tag{140}$$

We found that this simple expression leads to a correct near-minimum-weight design for the case of different allowable stresses in a structure, if the stresses in the elements at the optimum are equal to the allowable stresses in the elements.

The Newton-Raphson algorithm derived for the multiple-displacement constrained problem can also be derived for the stress-constrained problem by using the same procedure. The iterative relation would be

$$\{\lambda\}^{k+1} = \{\lambda\}^{k} - \phi[H]_k^{-1}\{g\}_k \tag{141}$$

where

$$H_{jp} = -\frac{1}{2}\sum_{i=1}^{n_1}\frac{R_{ij}R_{ip}}{\rho_i \ell_i A_i^3} \tag{142}$$

The optimality criterion for the stress constraints, Eq. 128, can also be written as

$$1 = \sum_{j=1}^{n}\frac{\lambda_j\{u\}_i^t[k]_i\{s^j\}_i}{A_i\rho_i\ell_i} \tag{143}$$

or

$$1 = \frac{\{u\}_i^t[k]_i\sum_{j=1}^{n}\lambda_j\{s^j\}_i}{\rho_i A_i \ell_i} \tag{144}$$

In Eq. 144, if the resultant displacements $\sum_{j=1}^{n}\lambda_j\{s^j\}_i$ associated with the ith element due to the virtual-load system are equal to $\{u\}_i$ owing

to the applied load, the optimality criterion becomes

$$1 = \frac{\{u\}_i^t [k]_i \{u\}_i}{\rho_i \ell_i A_i} \tag{145}$$

The assumption that $\sum_{j=1}^{n} \lambda_j \{s^j\}_i = \{u\}_i$ is equivalent to assuming that the product of the Lagrange multiplier and the virtual load applied to each element is equal to the force in the element owing to the load {P}. Eq. 145 can also be written as

$$1 = \frac{V_i}{\rho_i A_i \ell_i} = \frac{e_i}{\rho_i} \tag{146}$$

where V_i is the strain energy stored in the element and e_i is the strain energy density.

This criterion states that at the optimum the ratio of the strain energy density to the mass density is the same for all the elements. The criterion corresponds to a constraint on the generalized stiffness and can be derived by replacing the virtual load vector $\{S^1\}$ by the actual load vector {P} in Eq. 29.

A recurrence relation based on Eq. 146 can be written as

$$A_i^{k+1} = A_i^k \left(\frac{e_i}{\rho_i}\right)_k^{1/r} \tag{147}$$

This recurrence relation can be used as an approximation to Eq. 130 to solve a stress-constrained problem. In this case also if the maximum allowable stress in the different elements is not the same, it may not lead to a minimum-weight design. The advantage of Eq. 147 is that it is not required to calculate the virtual strain energies in the elements and the Lagrange multipliers. A linear form of Eq. 147 can be written as

$$A_i^{k+1} = A_i^k \left(1 + \frac{1}{r}\left(\frac{e_i}{\rho_i} - 1\right)\right)_k \tag{148}$$

SECTION V

ALGORITHM WITH THE RECIPROCAL DESIGN VARIABLE

In Sections 3 and 4 we have used the direct design variable A_i to define the objective function and the constraint equations. The recurrence relations and the equations to determine the Lagrange multipliers were expressed in terms of A_i. Now we will see how the optimality criterion and the algorithm is affected by defining the problem in terms of the reciprocal design variable z_i. The reciprocal design variable z_i and the direct design variable A_i are related by Eq. 9. We will derive the equations only for the multiple-displacement-constrained problem, since the conclusions from this can be readily extended to the stress-constrained problem.

OPTIMALITY CRITERION

The weight of the structure can be written as

$$W(z) = \sum_{i=1}^{n} \frac{\rho_i \ell_i}{z_i} \tag{149}$$

The constraint equations in terms of z_i can be written as

$$g_j = C_j - \overline{C}_j \leq 0 \qquad j=1,\ldots m \tag{150}$$

where

$$C_j = \sum_{i=1}^{n} Q_{ij} z_i \tag{151}$$

Using Eqs. 26 and 150, the Lagrangian can be written as

$$L(z, \lambda) = \sum_{i=1}^{n} \frac{\rho_i \ell_i}{z_i} + \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{m} Q_{ij} z_i - \overline{C}_j \right) \tag{152}$$

Differentiating this equation with respect to the design variable z_i and setting the result equal to zero gives the optimality criterion,

$$- \frac{\rho_i \ell_i}{z_i^2} + \sum_{j=1}^{m} \lambda_j Q_{ij} = 0 \tag{153}$$

or

$$1 = \frac{\rho_i \ell_i}{\sum_{j=1}^{m} \lambda_j Q_{ij} z_i^2} \qquad (154)$$

where $\lambda_j \geq 0$ and $\lambda_j g_j = 0$.

Eq. 154 can also be written as

$$1 = \frac{\rho_i}{\sum_{j=1}^{m} \lambda_j e_{ij}} \qquad (155)$$

where e_{ij} is the virtual strain energy density. Comparing the optimality criterion for the direct design variable in Eq. 66 with Eq. 155 shows them to be equivalent, even though one is the reciprocal of the other.

RECURRENCE RELATIONS

Applying the same logic used to write the exponential recurrence relation for A_i, (Eq. 70), the recurrence relation for z_i can be written by using the optimality criterion in Eq. 154. This gives

$$z_i^{k+1} = z_i^k \left(\sum_{j=1}^{m} \frac{\lambda_j Q_{ij} z_i^2}{\rho_i \ell_i} \right)_k^{-1/r} \qquad (156)$$

This recurrence relation is equivalent to Eq. 70 since one can be obtained from the other by using Eq. 9.

The linear recurrence relation for the reciprocal design variable can be obtained by expanding Eq. 156 using the binomial theorem and retaining only the linear terms. This gives

$$z_i^{k+1} = z_i^k \left(1 - \frac{1}{r} \left(\sum_{j=1}^{m} \frac{\lambda_j Q_{ij} z_i^2}{\rho_i \ell_i} - 1 \right) \right)_k \qquad (157)$$

This equation can be expressed in terms of A_i by using Eq. 9. This gives

$$A_i^{k+1} = A_i^k \left(- \frac{1}{r} \left(\sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} - 1 \right) \right)_k^{-1} \qquad (158)$$

Comparing this equation with the linear recurrence relation for A_i in Eq. 71 shows that they are not equivalent.

EVALUATION OF THE LAGRANGE MULTIPLIERS

The set of equations to determine the Lagrange multipliers can be obtained by considering a change in the constraint owing to a change in the design variable z_i. The change in a constraint can be written as

$$\Delta g_j = g_j\,(z + \Delta z) - g_j(z) \tag{159}$$

$$= \sum_{i=1}^{n} \frac{\partial g_j}{\partial z_i}\,\Delta z_i \tag{160}$$

Using Eqs. 150 and 157 and remembering that $g_j(z+\Delta z)=0$, Eq. 160 reduces to

$$\sum_{p=1}^{m} \lambda_p^{k+1} \left(\sum_{i=1}^{m} \frac{Q_{ij} Q_{ip} z_i^3}{\rho_i \ell_i} \right) = r(C_j^k - \overline{C}_j) + \sum_{i=1}^{n} (Q_{ij} z_i)_k \tag{161}$$

Comparing this relation with Eq. 84 shows that they are equivalent and that one can be obtained from the other by using the relationship between A_i and z_i. Similarly one can show that the iterative relations for the Newton-Raphson method for z_i are also equivalent to those of the direct-design variable A_i (Eq. 101).

The equations equivalent to Eq. 156 and 157 for the stress-constrained problem can be written as

$$z_i^{k+1} = z_i^k \left(\sum_{j=1}^{n} \frac{\lambda_j R_{ij} z_i^2}{\rho_i \ell_i} \right)_k^{-1/r} \tag{162}$$

$$z_i^{k+1} = z_i^k \left(1 - \frac{1}{r} \left(\sum_{j=1}^{n} \lambda_j \frac{R_{ij} z_i^2}{\rho_i \ell_i} - 1 \right) \right)_k \tag{163}$$

The equations derived in this section show that the definition of the problem in terms of the reciprocal design variable affects the linear recurrence relation. However, the optimality criterion, the exponential recurrence relation and the equation to determine the Lagrange multipliers are not changed so as to affect the behavior of the algorithm.

SECTION VI

SYSTEM STABILITY CONSTRAINT

The constraint equation for the linear static buckling of a structure can be written as

$$g_j = \mu_j - \bar{\mu} \tag{164}$$

where $\bar{\mu}$ is the lowest critical load factor and μ_j is given by Eq. 16. The Lagrangian for the constraint on static stability can be written as

$$L(A, \lambda) = \sum_{i=1}^{n} \rho_i A_i \ell_i + \sum_{j=1}^{m} \lambda_j (\mu_j - \bar{\mu}) \tag{165}$$

The optimality criterion is obtained by differentiating Eq. 165 with respect to A_i and setting the result equal to zero. This gives

$$\rho_i \ell_i + \sum_{j=1}^{m} \lambda_j \frac{\partial \mu_j}{\partial A_i} = 0 \tag{166}$$

The gradient of μ_j can be found by writing Eq. 15 as

$$\{\eta\}_j^t [K] \{\eta\}_j + \mu_j \{\eta\}_j^t [K_G] \{\eta\}_j = 0 \tag{167}$$

and differentiating it with respect to A_i. This gives

$$\begin{aligned} \frac{\partial \mu_j}{\partial A_i} \left[\{\eta\}_j^t [K_G] \{\eta\}_j \right] = &- \frac{1}{A_i} \{\eta\}_j^t [k]_i \{\eta\}_j - \mu_j \{\eta\}_j^t \left[\frac{\partial}{\partial A_i} [K_G] \right] \{\eta\}_j \\ &- 2 \frac{\partial}{\partial A_i} \{\eta\}_j^t \left[[K]\{\eta\}_j + \mu_j [K_G] \{\eta\}_j \right] \end{aligned} \tag{168}$$

The second and third term on the right side of this equation are equal to zero, and therefore we can write

$$\frac{\partial \mu_j}{A} = - \frac{1}{A_i} \frac{\{\eta\}_k^t [k]_i \{\eta\}_j}{\{\eta\}_j^t [K_G] \{\eta\}_j} \tag{169}$$

Substituting this equation in Eq. 166, the optimality criterion can be written as

$$1 = \sum_{j=1}^{m} \bar{\lambda}_j \frac{\bar{Q}_{ij}}{\rho_i A_i^2 \ell_i} \tag{170}$$

where

$$\bar{Q}_{ij} = A_i \{\eta\}_j^t \, [k]_i \{\eta\}_j \tag{171}$$

and

$$\bar{\lambda}_j = \frac{\lambda_j}{\{\eta\}_j^t [K_G] \{\eta\}_j} \tag{172}$$

Comparing Eqs. 170 with Eq. 65 shows that the optimality criterion for the system-stability constraint problem has the same form as the optimality criterion for the displacement-constraint problem. Eq. 170 can also be written as

$$1 = \sum_{j=1}^{m} \bar{\lambda}_j \frac{q_{ij}}{\rho_i} \tag{173}$$

where

$$q_{ij} = \frac{\bar{Q}_{ij}}{A_i^2 \ell_i} \tag{174}$$

is the energy density in an element in the buckled mode. The exponential recurrence relation and the linear recurrence relation for the system-stability constraint problem are given by

$$A_i^{k+1} = A_i^k \left(\sum_{j=1}^{m} \bar{\lambda}_j \frac{\bar{Q}_{ij}}{\rho_i \ell_i A_i^2} \right)_k^{1/r} \tag{175}$$

and

$$A_i^{k+1} = A_i^k \left(1 + \frac{1}{r} \left(\sum_{j=1}^{m} \bar{\lambda}_j \frac{\bar{Q}_{ij}}{\rho_i \ell_i A_i^2} - 1 \right) \right)_k \tag{176}$$

Equations to evaluate the Lagrange multipliers λ_j can be written by using the procedure discussed for the multiple-displacement-constrained problem.

SECTION VII

EFFECT OF STRUCTURAL SYMMETRY ON THE ALGORITHM

In the previous sections when deriving the optimality criterion and developing the algorithm, we had not taken into consideration the possible symmetry of the structure. The symmetry may be due to the nature of the constraints imposed on the structure and also due to the multiple-loading conditions. We had mentioned that the multiple-loading conditions can be treated as multiple constraints. A substantial amount of computational effort can be saved by making use of the symmetry of the structure in the analysis phase and particularly in the optimization phase of the algorithm. A modification to the optimization algorithm becomes particularly essential when the structure is idealized with a large number of elements, the number of loading conditions is more than one, and the number of active constraints is large.

We will consider a very simple case and illustrate what modifications to the algorithm are required. Consider a symmetric structure subjected to multiple-displacement constraints. Let us assume that there are five constraints which are active at the kth iteration. The exponential recurrence relation can be written as

$$A_i^{k+1} = A_i^k \left(\sum_{j=1}^{5} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} \right)^{1/r} \tag{177}$$

Now because of the symmetry of the constraints let us assume that $\lambda_1 = \lambda_5 = \tilde{\lambda}_1$ and $\lambda_2 = \lambda_4 = \tilde{\lambda}_2$ and $\lambda_3 = \tilde{\lambda}_3$. The recurrence relation (Eq. 177) can now be written as

$$A_i^{k+1} = A_i^k \left(\tilde{\lambda}_1 \left(\frac{Q_{i1} + Q_{i5}}{\rho_i \ell_i A_i^2} \right) + \tilde{\lambda}_2 \left(\frac{Q_{i2} + Q_{i4}}{\rho_i \ell_i A_i^2} \right) + \tilde{\lambda}_3 \frac{Q_{i3}}{\rho_i \ell_i A_i^2} \right)^{1/r} \tag{178}$$

The linear equations (Eq. 85) to determine the Lagrange multipliers can be written as

$$[Q^*]_{5x5} \{\lambda\}_5 = \{C^*\}_5 \tag{179}$$

where Q^*_{ij} are the coefficients of the square matrix multiplying the λ vector in Eq. 85 and C* is the right side of Eq. 85. These quantities are evaluated for five constraints. In Eq. 179 $Q^*_{ij} = Q^*_{ji}$. Replacing the λ's by the α's, and remembering that in Eq. 179 because of symmetry the first equation is the same as the fifth equation and the second equation is the same as the fourth equation, we can write

$$\begin{aligned}
\alpha_1 (Q^*_{11} + Q^*_{15}) + \alpha_2(Q^*_{12} + Q^*_{14}) + \alpha_3(Q^*_{13}) &= C^*_1 \\
\alpha_1 (Q^*_{21} + Q^*_{25}) + \alpha_2(Q^*_{22} + Q^*_{24}) + \alpha_3(Q^*_{23}) &= C^*_2 \\
\alpha_1 (Q^*_{31} + Q^*_{35}) + \alpha_2(Q^*_{32} + Q^*_{34}) + \alpha_3(Q^*_{33}) &= C^*_3
\end{aligned} \tag{180}$$

These three equations are not symmetric. Taking into consideration the symmetry of the constraints on the coefficients Q^*_{ij}, Eq. 180 can be written as

$$\begin{aligned}
(Q^*_{11} + Q^*_{15})\alpha_1 + 2Q^*_{12}\alpha_2 + Q^*_{13}\alpha_3 &= C^*_1 \\
2Q^*_{12}\alpha_1 + (Q^*_{22} + Q^*_{24})\alpha_2 + Q^*_{23}\alpha_3 &= C^*_1 \\
2Q^*_{13}\alpha_1 + 2Q^*_{23}\alpha_2 + Q^*_{33}\alpha_3 &= C^*_3
\end{aligned} \tag{181}$$

Thus because of the symmetry we have to solve three equations instead of five. In addition Eq. 179 would require evaluation of fifteen Q^*_{ij} coefficients as compared with eight coefficients for Eq. 181. For this small problem this modification does not appear to be a substantial reduction in the computational effort. However, for a structure with a large number of design variables and potentially active constraints, it would save considerable CP time. The modification to the algorithm discussed here can be extended to other types of constraints, to other symmetric conditions, and also to other algorithms discussed previously.

Note that if in a structure all the 'p' Lagrange multipliers associated with all the active constraints are equal, because of the symmetry, all 'p' equations will reduce to a single equation. This will require evaluation of only one Lagrange multiplier. Thus an apparently multiple-constraint problem is reduced to an equivalent single-constraint

problem. For such a problem an algorithm based on the single dominant constraint would give the same design as obtained by using the multiple-constraint algorithm. The 72-bar tower problem considered in References 1, 2, 29, and other references fall under this category.

SECTION VIII

OPTIMIZATION PROCEDURE

In Sections III through VII we have derived the optimality criterion, the different recurrence relations to modify the design variables, and the equations to estimate the Lagrange multipliers. We will discuss in this section how different algorithms can be developed based on these equations and what options we have in selecting an algorithm.

The main steps in the optimization of a structure are:

(1) Assign initial sizes to all the elements. For all example problems we will assume them equal.

(2) Analyze the structure using the displacement method of analysis.

(3) Determine the stresses in the elements.

(4) Scale the design to satisfy all the constraints. The scaled design is a feasible design. After scaling, at least one constraint will be on the constraint surface.

(5) Separate the constraints into potentially active and passive categories.

(6) Separate the elements into active and passive categories.

(7) Determine the flexibility coefficients corresponding to the potentially active constraints.

(8) Evaluate the Lagrange multipliers.

(9) Modify the design variables using a recurrence relation.

(10) Terminate the iterations if the specified criterion is satisfied or go to step 2. The terminating criterion may be the number of iterations or the percentage difference in the weight of the structure between the consecutive iterations.

The first four steps are common to all the algorithms. The subsequent steps differ depending upon (a) the division of active and passive constraints, (b) the value of the parameters selected to control the step size, (c) the method used to determine the Lagrange multipliers, (d) the recurrence relation used to modify the design variables, and (e) the number of times the design variables are modified before reanalyzing the structure. We will discuss in detail the different options available under these categories.

(a) Active and Passive Constraints

The active constraints or potentially active constraints are those that are included in the constraint equations. The flexibility coefficients and the Lagrange multipliers associated with the active constraints enter the optimality criterion and the recurrence relation. The remaining constraints which are imposed on the structure are called passive constraints, and these are satisfied by the simple approach of scaling the individual design variables. In the case of a displacement and stress constraint problem, we can use one of the following approaches to design a structure.

1) Treat all the displacements as well as the stress constraints as potentially active constraints, determine the Lagrange multipliers associated with all the active constraints, and use them in the recurrence relation. In this case only the elements whose sizes are governed by minimum or maximum size constraints would be passive.

2) Consider only the displacement constraints as potentially active and all the stress constraints as passive. Here, the stress constraints are not directly included in the constraint equations. Elements in which the stresses are greater than the maximum allowable stress are sized by using the FSD algorithm. Thus the size of an element is a maximum of the sizes obtained by (1) the recurrence relation for the displacement constraints, (2) the FSD algorithm, or (3) the minimum size constraint. Any element whose size is not equal to the one obtained by using a recurrence relation is characterized as a passive element. Since the number of elements and which elements are passive, affect the equations used to

determine the Lagrange multipliers and consequently the recurrence relation, one must keep track of the passive elements and update the list whenever any change occurs.

3) Treat only one or some dominant displacement constraints as potentially active and consider the remaining displacement constraints and all the stress constraints as passive.

4) Treat all the displacement constraints as being independent of each other, and the stress constraints as passive constraints. In this case the size of an element is a maximum of the sizes obtained by (1) the recurrence relation for the single displacement constraint applied to each displacement constraint separately, (2) the FSD algorithm, or (3) the minimum size constraint. This method is known as the envelope method.

For a structure subjected to stress constraints only, we can use one of the following approaches:

1) Treat all the stress constraints as potentially active, determine the Lagrangian multipliers associated with the active constraints, and use them in the recurrence relation. The passive elements in this case will be the elements whose sizes are governed by the minimum size limits.

2) Treat all the stress constraints as active, but approximate the virtual load system with the actual load system. In this case the Lagrange multipliers are not determined. The elements are sized on the basis of the strain energy stored in the element due to the applied load.

3) Treat all the stress constraints as active, but use the FSD algorithm.

(b) Step Size

The convergence behavior depends on the parameter 'r' in the recurrence relation and also the equations used to determine the Lagrange multipliers and the parameter 'b' in the Newton-Raphson iterative method. We have derived a relationship between b, ϕ, and r. For structural

optimization, r=2 is generally adequate. However for some problems, depending on the behavior of the constraints, 'r' must be increased to reduce the step size and get better convergence. The parameter 'r' can be kept constant through all the iterations or, based on certain criteria, can be changed after each iteration. A good indication of the proper selection of 'r' is a reduction in the weight of the structure after each iteration. Another approach in controlling the step size is to generate an intermediate design vector whenever the weight of structure is more than the previous iteration. This can be done by taking the average of the design variables in the previous and the present iteration. For example, if the weight of the structure with the design variables A_i^{k+1} is greater than the one with A_i^k, the intermediate design variable would be $(A_i^{k+1} + A_i^k)/2$. This process can be continued until one obtains a design with a weight less than the weight of the previous iteration.

(c) Lagrange Multipliers

When the structure is subjected to inequality constraints, we use only those Lagrange multipliers which are positive. A negative Lagrange multiplier is permissible only if the specific constraint is to be enforced as an equality constraint. The Lagrange multipliers can be determined by one of the following methods:

1) Linear simultaneous equation.
   Eqs. 86, 87 for the displacement constraints.
   Eq. 136 for the stress constraints.

2) Newton-Raphson iteration.
   Eq. 101 for the displacement constraints.
   Eq. 141 for the stress constraints.

3) Exponential and linear recurrence relations.
   Eqs. 74 and 76 for the displacement constraints.
   Eqs. 132 and 133 for the stress constraints.

4) Approximate relations.
   Eq. 95 for the displacement constraint.
   Eqs. 138, 139, 140 for the stress constraints.

Under the last category we can include a number of degenerate equations obtained from the first method. For the first two cases and some equations of the fourth case, the coefficients must be calculated Q_{ij} or R_{ij} before the Lagrange multipliers can be determined. In the third case the Lagrange multipliers are estimated based on the ratio of the actual value of the constraint to the limiting value of the constraint, and the coefficients Q_{ij} or F_{ij} need not be known before the λ's are estimated. The advantages and the disadvantages of using these methods were discussed in the section where they were derived.

(d) Recurrence Relations

In the case of the displacement constraint problem there are basically three distinct relations. These are:

1) Exponential relation for A_i or z_i (Eq. 70 or 156).
2) Linear relation for A_i (Eq. 71 or 72).
3) Linear relation for z_i (Eq. 157 or 158).

We can use any one of these relations. In these relations we can keep the step-size parameter 'r' constant or change it after each iteration. All these relations need the evaluation of the Lagrange multipliers.

For the stress constraint problem we can use one of the following:

1) Exponential relation for A_i or z_i (Eq. 130 or 162).
2) Linear relation for A_i (Eq. 131).
3) Linear relation for z_i (Eq. 163).
4) FSD algorithm (Eq. 117).
5) Exponential relation for A_i based on the strain energy distribution (Eq. 147).
6) Linear relation for A_i based on the strain energy distribution (Eq. 148).

The first three of these relations require evaluation of the Lagrange multipliers before they can be used. The fourth requires knowing the force in the element, and the last two use the strain energy stored in

an element. For the last three the Lagrange multipliers are not evaluated, only the information from the applied load vector is used.

e) The flexibility coefficients Q_{ij} used to evaluate the Lagrange multipliers and modify the design variables are calculated with the assumed values of the design variables. When we modify the design variables, the previously calculated flexibility coefficients are no longer valid. We have two choices. One is to reanalyze the structure and calculate the new flexibility coefficients. And the other choice is to assume that the flexibility coefficients are not substantially affected with small changes in the design variables and reuse them. The second option would require a smaller number of analyses. However, note that the design does not move away from the region where the flexibility coefficients are no longer valid.

SECTION IX

ILLUSTRATIVE PROBLEMS

This section presents results for the sample problems which are solved by using some of the algorithms and the optimization procedure discussed in the previous sections. Some of the problems selected for presentation are standard problems and have been used by several investigators to study the performance of their algorithms. The first problem is a 10-bar truss. This structure is designed to satisfy stress-displacement constraints and different allowable stresses in the elements. The second structure is a 200-bar truss subjected to 5 loading conditions. This structure is designed with constraints on stresses and displacements. The third structure is a cantilever box-beam with the top and bottom skins made of a layered composite material. The box-beam is designed to satisfy stress constraints and local plate buckling. The fourth structure is a composite wing structure designed to satisfy stress constraints in all the elements and a twist constraint at the free end. The last structure is a three-dimensional dome designed to satisfy the constraint on system stability. For all the problems, unless otherwise mentioned, the sizes of all the elements are equal for the first iteration. The CP time given for some problems is for the CDC CYBER 74/175.

In problems where the linear equations (Eqs. 86 or 136) are solved to determine the Lagrange multipliers, the equations were first arranged in the order of the degree of activity of the constraints. The first equation corresponds to the constraint that has been closest to the constraint surface, and the last equation corresponds to the constraint that was farthest from the constraint surface. The rearranged equations were solved by an iterative method. In the first iteration, only one equation was solved, assuming that there was only one constraint, and during the subsequent iterations a new equation was added each time. This process was continued until all the equations corresponding to all the potentially active constraints were solved.

EXAMPLE 1. TEN-BAR TRUSS (DISPLACEMENT AND STRESS CONSTRAINTS)

The truss shown in Figure 1 was designed to satisfy a stress limit of 25 ksi in all the elements and a displacement limit of ± 2.0 in. at all the node points in the x and y direction.

Case I

The truss is designed by considering stress constraints as well as displacement constraints as potentially active constraints. The three recurrence relations used to modify the design variables were (1) Case Ia, the exponential relation for A_i or z_i (Eqs. 70, 130, 156, 162), (2) Case Ib, the linear relation for A_i (Eqs. 71 and 131), (3) Case Ic, the linear relation for z_i (Eqs. 157 and 163). The Lagrange multipliers for

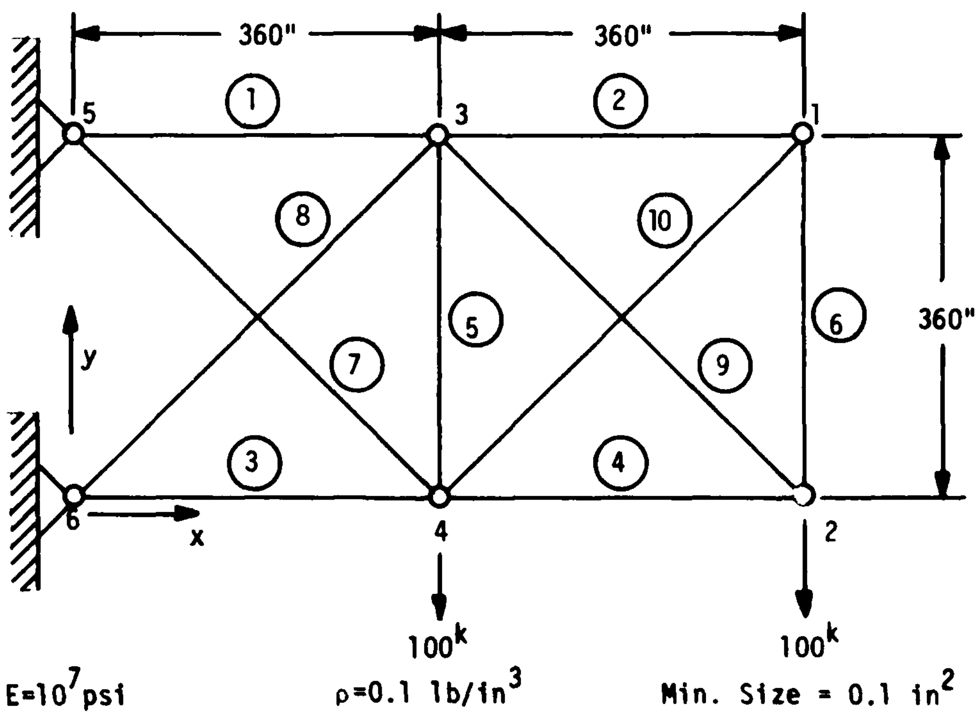


Figure 1. Ten-Bar Truss

the three cases are determined by solving the linear simultaneous equations given in Eqs. 86 and 136. The potentially active constraints were determined at each iteration. For the first iteration, all the constraints satisfying the conditions $C_j/\overline{C}_j = \delta = 0.5$ for the scaled design were assumed to be potentially active. The maximum value of $C_j/\overline{C}_j$ is equal to unity for the scaled design. In subsequent iterations, $C_j/\overline{C}_j$ was determined based upon the activity of the constraints in the previous iteration. For example, δ for the (k+1)th iteration is equal to $C_j/\overline{C}_j$ of the kth iteration that includes all the active constraints. In addition a check was made to make sure that there was at least one nonactive constraint included in the potentially active constraints. This was essential in order to assure that all the active constraints were used in the recurrence relation. The step-size parameter 'r' for all three cases was set equal to 2. The structure was analyzed after each iteration.

The iteration history for Cases Ia through Ic is given in Table 1. The table also contains the active constraints at each iteration and the total time needed to complete the iterations for Case Ic. For Case Ia, after the 12th iteration, the weight of the structure was found to oscillate at weights greater than 5070.1 lb. This behavior was due to the large step size. If the step size parameter 'r' had been increased after the 12th iteration, in order to reduce the step size, the additional iterations would lead to a design with a weight of 5060.85 lb. The active constraints at the optimum for this design were the vertical displacement at node 1 and the stress in member 5. For Case Ia, at the 10th iteration, the scaled weight of the structure jumped to 6069.3 lb, because the stress in element 5 suddenly became active. The iterations for Cases Ib and Ic lead to a design with a minimum weight of 5076.6 lb with the vertical displacements at nodes 1 and 3 being active and equal to 2.0 in. The stress constraint for Cases Ib and Ic did not become active. Both of the designs satisfy the optimality criterion. The design with a weight of 5076.6 lb is a relative minimum. The iteration history for the three cases is shown in Figure 2. The computer time shown in this figure is for Case Ic. The total time required to complete 12 iterations for Case Ia and 14 iterations for Case Ib was 1.78 and 1.88 seconds, respectively.

TABLE 1

ITERATION HISTORY FOR 10-BAR-TRUSS
STRESS AND DISPLACEMENT CONSTRAINTS - CASE I

	Case Ia Exponential Relation		Case Ib Linear Relation - A_i		Case Ic Linear Relation - z_i		
Iteration No.	Weight	Active Constraints	Weight	Active Constraints	Weight	Active Constraints	Total Time in Seconds
1	8266.1	5	8266.1	5	8266.1	5	0.09
2	6034.3	5	6666.8	5	6911.9	5	0.21
3	5824.3	5	6196.3	5	6438.4	5	0.33
4	5691.6	5	5946.9	5	6194.4	5	0.44
5	5574.6	5	5759.1	5	6018.7	5	0.55
6	5448.0	5	5627.3	5	5876.9	5	0.67
7	5305.7	2,5	5516.0	5	5749.5	5	0.78
8	5191.0	2,5	5400.5	5	5627.7	5	0.89
9	5083.5	2	5278.2	2,5	5510.6	5	1.01
10	6069.3	2,5,5*	5185.1	2 5	5401.5	5	1.13
11	5087.8	2	5118.3	2 5	5305.6	2,5	1.27
12	5070.1	-	5091.2	2 5	5228.0	2,5	1.41
13			5078.2	2,5	5169.2	2,5	1.54
14			5076.6	2,5	5134.1	2,5	1.68
15					5095.3	2,5	1.81
16					5085.0	2,5	1.95
17					5078.4	2,5	2.08
18					5076.6	2,5	2.23

Note: Active Constraints 2 Vertical Displacement at Node 1

5 Vertical Displacement at Node 2

5* Element Number 5

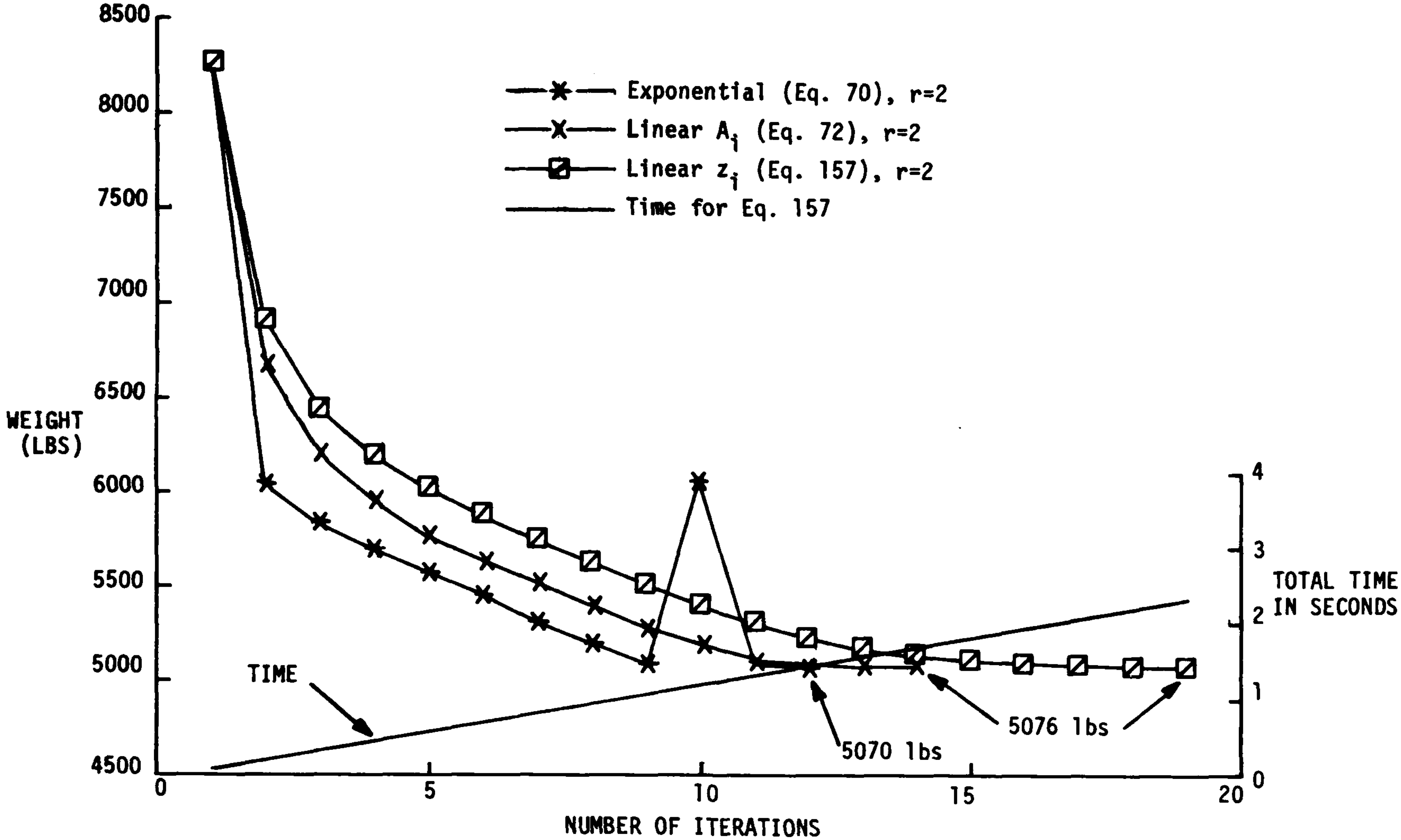


Figure 2. Iteration History for 10-Bar-Truss Stress-Displacement Constraints - Case I

Case II

The truss is designed by using the exponential recurrence relation with the step-size parameter 'r' set equal to 4 after each analysis of the structure. The Lagrange multipliers were determined by using Eqs. 86 and 136. For Case IIa, the structure was reanalyzed after each iteration. For Case IIb, a maximum of 10 subiterations were allowed before reanalyzing the structure with a criterion set to satisfy the constraints equal to 10^{-7}. During the subiterations the step-size parameter was doubled, to reduce the step size, whenever the scaled weight of the structure was greater than the previous lower-weight design within the subiterations. We have referred to this approach in the last seciton as the modified Newton-Raphson approach. The iteration history and the total time needed to complete the specified number of iterations are given in Table 2. Case IIa and Case IIb gave minimum weight designs of 5076.6 lb and 5060.8 lb respectively. However, the approach of using repeated subiterations (Case IIb) needed substantially more computer time.

TABLE 2

ITERATION HISTORY OF 10-BAR-TRUSS
STRESS-DISPLACEMENT CONSTRAINTS - CASE II

	Case IIa		Case IIb	
Iteration No.	Weight	Total Time in Seconds	Weight	Total Time in Seconds
1	8266.1	0.09	8266.1	0.43
2	6423.2	0.21	6017.7	0.92
3	6009.5	0.34	5818.9	1.39
4	5856.1	0.45	5689.0	1.82
5	5780.1	0.58	5572.6	2.36
6	5719.4	0.70	5445.3	2.90
7	5660.4	0.81	5303.1	3.50
8	5599.3	0.93	5198.9	4.16
9	5536.6	1.04	5105.3	4.95
10	5469.9	1.16	5079.6	5.59
11	5398.4	1.23	5074.3	6.59
12	5320.4	1.41	5064.0	7.21
13	5232.3	1.56	5062.4	7.96
14	5155.0	1.69	5061.2	8.17
15	5076.9	1.84	5061.0	9.46
16	5076.6	2.00	5060.8	10.20

The iteration history for Case II is shown in Figure 3. The Lagrange multipliers and the areas of the members for the two minimum-weight designs are given in Table 3, and are designated as Design 1 and Design 2.

EXAMPLE 2. TEN-BAR TRUSS (STRESS CONSTRAINTS)

The truss shown in Figure 1 was designed to satisfy different stress constraints in the elements.

Case I

The maximum allowable stress for all the elements was 25 ksi except the maximum allowable stress was increased to 50 ksi for element 9. The FSD design for this case is not the correct minimum-weight design. The minimum weight obtained by FSD algorithm was 1725 lb.

The Lagrange multipliers for this case were obtained by solving Eq. 136 and all 10 stress constraints were assumed to be potentially active in all the iterations. The structure was reanalyzed after each iteration. The parameter 'r' was set equal to 2 for all three recurrence relations: the exponential relation (Eq. 130), the linear relation for A_i (Eq. 131) and the linear relation for z_i (Eq. 163). The parameter r=2 was found to be too large to obtain proper convergence for the exponential relation. Therefore, these results are not given here. The iteration history for the other two cases, Case Ia (Eq. 131) and Case Ib (Eq. 163), is given in Table 4. This table contains the active constraints at each iteration and the total time required to complete the specified number of iterations. The iteration history is shown in Figure 4. The minimum-weight design for this problem is 1497.6 lb with the stresses in elements 1, 2, 3, 4, 6, 7, 8, 10 equal to 25 ksi. The stress at the optimum in element 9 is 37.5 ksi, and this is not an active constraint. The cross-sectional areas of all the elements and the Lagrange multipliers associated with the minimum-weight design are given in Table 3 as Design 3.

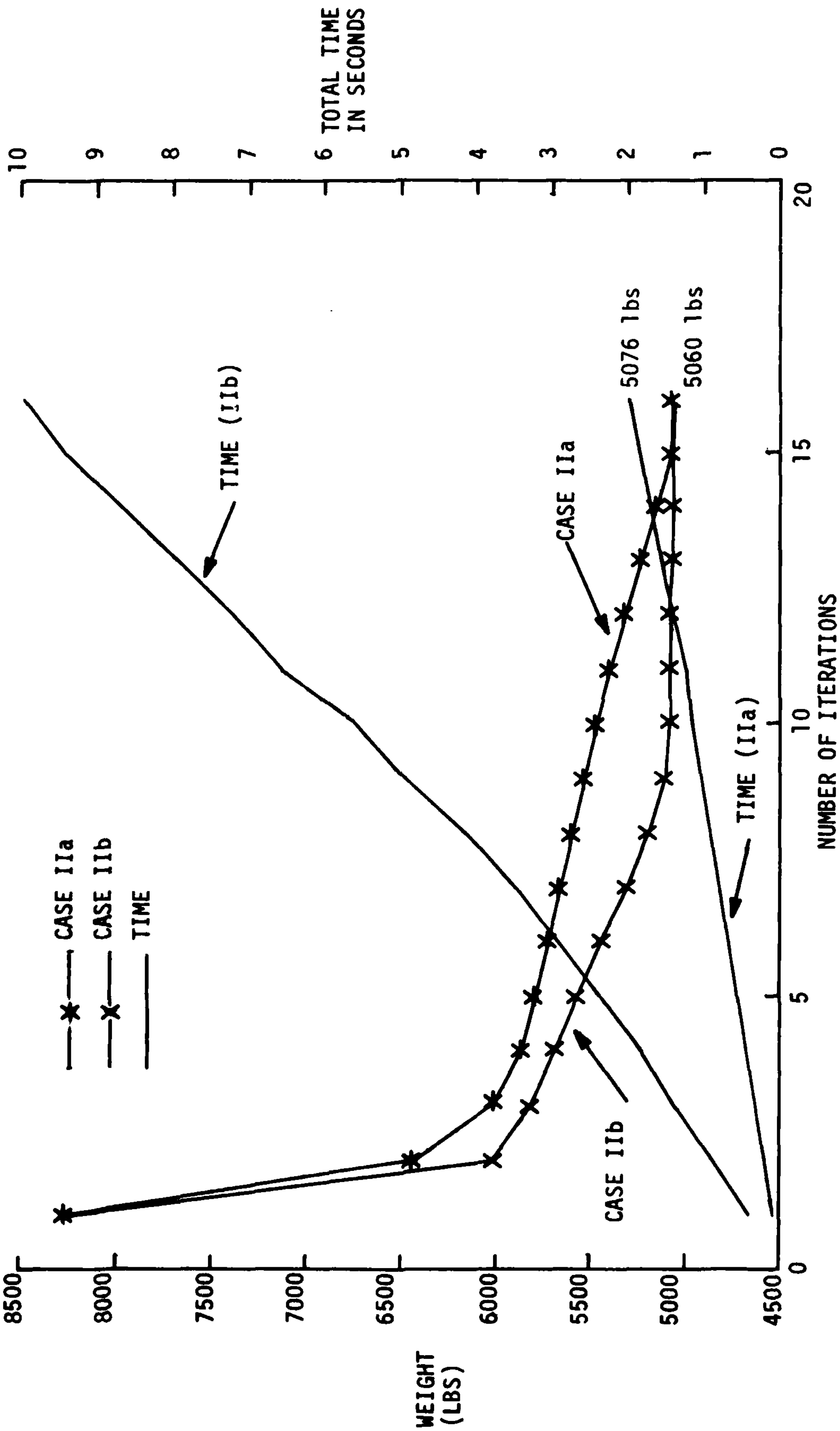


Figure 3. Iteration History for 10-Bar-Truss Stress-Displacement Constraints - Case II

TABLE 3

MINIMUM-WEIGHT DESIGNS OF 10-BAR TRUSS

Design 1		Design 2		Design 3		
Member	Area	Member	Area	Member	Area	Lagrange Multiplier
1	30.7297	1	30.5210	1	7.9000	246.66
2	0.1000	2	0.1000	2	0.1000	69.99
3	23.9407	3	23.1999	3	8.1000	393.33
4	14.7331	4	15.2229	4	3.9000	86.66
5	0.1000	5	0.1000	5	0.1000	0.0
6	0.1000	6	0.5514	6	0.1000	69.99
7	8.3406	7	7.4572	7	5.7983	329.98
8	20.9510	8	21.0364	8	5.5154	122.56
9	20.8358	9	21.5284	9	3.6770	0.0
10	0.1000	10	0.1000	10	0.1414	103.70
Weight	5076.66	Weight	5060.85	Weight	1497.6	

Active Constraints	Y-deflection at Node 1	Y-deflection at Node 2	Y-deflection at Node 1	Element 5
Lagrange Multiplier	1638.0	885.3	2440.66	136.95

Note: Elements with zero Lagrange multipliers are inactive constraints.

TABLE 4

ITERATION HISTORY FOR 10-BAR-TRUSS
STRESS CONSTRAINTS - CASE I

Iteration No.	Case Ia: Linear Relation - A_i			Case Ib: Linear Relation - z_i		
	Weight	Active Constraints	Total Time in Seconds	Weight	Active Constraints	Total Time in Seconds
1	3434.9	1,3,7	0.18	3434.9	1,3,7	0.19
2	2444.3		0.36	2541.7		0.38
3	1835.7	1 2,3,4	0.61	2103.3		0.57
4	2144.5	6 7,10	0.78	1804.6	1,2,3,4	0.79
5	1796.1		0.97	1686.5	6,7,10	1.00
6	1758.7		1.15	1637.9		1.18
7	1591.1		1.35	1594.4		1.38
8	1556.2	1,2,3,4	1.55	1572.9	1,2,3,4	1.58
9	1532.5	6,7,8,10	1.75	1558.3	6,7,8,10	1.77
10	1524.7		1.96	1547.1		1.96
11	1518.9		2.12	1537.9		2.16
12	1514.2		2.39	1530.3		2.35
13	1510.2		2.62	1524.6		2.55
14	1506.3		2.35	1520.3		2.76
15	1504.0		3.07	1516.2		2.97
16	1501.6		3.32	1512.7		3.20
17	1499.7		3.57	1509.6		3.42
18	1498.1		3.84	1506.9		3.65
19				1504.5		3.86
20				1502.5		4.09
21				1500.3		4.31
22				1499.3		4.55

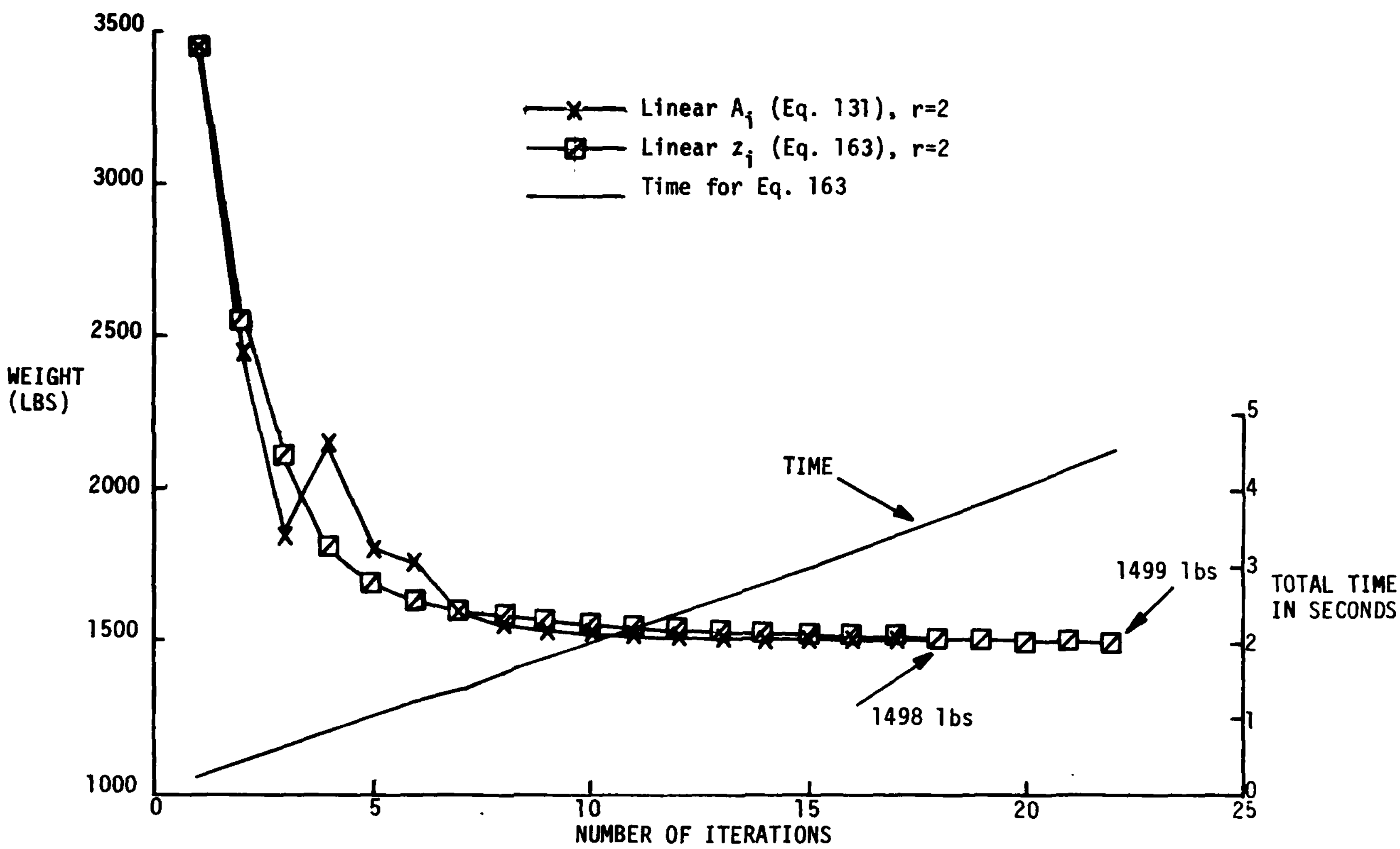


Figure 4. Iteration History for 10-Bar-Truss Stress Constraints - Case I

Case II

The problem solved was the same as Case I above, except that the Lagrange multipliers were determined by using the recurrence relation in Eq. 132. The step-size parameter r was set equal to 2, and the parameter 'b' in Eq. 132 was set equal to 0.5 (See Eq. 98). The three recurrence relations used to modify the design variables were the exponential relation (Eq. 130), the linear relation for A_i (Eq. 131) and the linear relation for z_i (Eq. 163). The Lagrange multipliers for the first iteration were assumed to be proprotional to the forces in the bars. All Lagrange multipliers were normalized so that the maximum value of the Lagrange multiplier was equal to unity. The structure was reanalyzed each time the design variables were modified using the recurrence relations. The exponential relation gave a minimum weight design of 1500.6 lb after 13 iterations. (See Table 5.) Use of both linear relations caused oscillations in the weight of the structure and did not give a good minimum-weight design. Therefore the problem was resolved by using the three recurrence relations. However, in the first iteration the design variables. However, in the first iteration the design variables were modified by using the FSD algorithm (Eq. 117). The iteration history for the three cases, IIa (Eq. 130), IIb (Eq. 131), and IIc (Eq. 163) is given in Table 5. The table also contains the total computer time required to complete the specified number of iterations. This time is substantially less than that given in Table 4, where Eq. 136 was used to determine the Lagrange multipliers. The iteration history for the three cases is shown in Figure 5. Previous experience with Eq. 132 has shown that with the exponential recurrence relation with r=2 and the parameter b increased to 4 to reduce the step size, a design with a weight 1497.6 lb can be obtained. However, this needs a substantially larger number of iterations.

Case III

The 10-bar truss with stress constraints was solved with a maximum allowable stress in element 9 equal to 37.5 ksi and a maximum allowable stress in all other elements equal to 25 ksi. The stress in element 9 was selected on the basis that its stress does not increase beyond 37.5 ksi for the minimum-weight design. The structure was designed by using

TABLE 5

ITERATION HISTORY FOR 10-BAR-TRUSS
STRESS CONSTRAINTS - CASE II

Iteration No.	Case IIa Exp. Relation	Case IIa Exp. Relation	Case IIb Linear Relation - A_i	Case IIc Linear Relation - z_i	Total Time in Seconds
1	3434.9	3434.9	3434.9	3434.9	0.08
2	1969.1	1812.7*	1812.7*	1812.7*	0.16
3	1636.5	1942.1	1830.6	1801.6	0.24
4	1673.1	1668.1	1673.6	1734.0	0.35
5	1653.9	1678.3	1687.8	1633.5	0.40
6	1557.3	1600.5	1734.9	1742.9	0.48
7	1553.6	1567.3	1625.5	1774.1	0.56
8	1538.4	1554.0	1551.3	1631.7	0.64
9	1526.0	1537.7	1542.8	1597.1	0.72
10	1516.6	1525.6	1532.1	1559.7	0.79
11	1509.4	1516.1	1521.5	1553.5	0.88
12	1504.1	1509.3	1514.3	1530.9	0.96
13	1500.6	1505.7	1509.2	1523.0	1.05
14		1502.4	1505.8	1515.6	1.13
15		1500.7	1504.2	1514.1	1.21
16		1499.3	1503.2	1512.6	1.29
17			1502.0	1510.6	1.37
18				1508.7	1.46
19				1506.3	1.54
20				1505.0	1.61
21				1503.2	1.70

* FSD Design

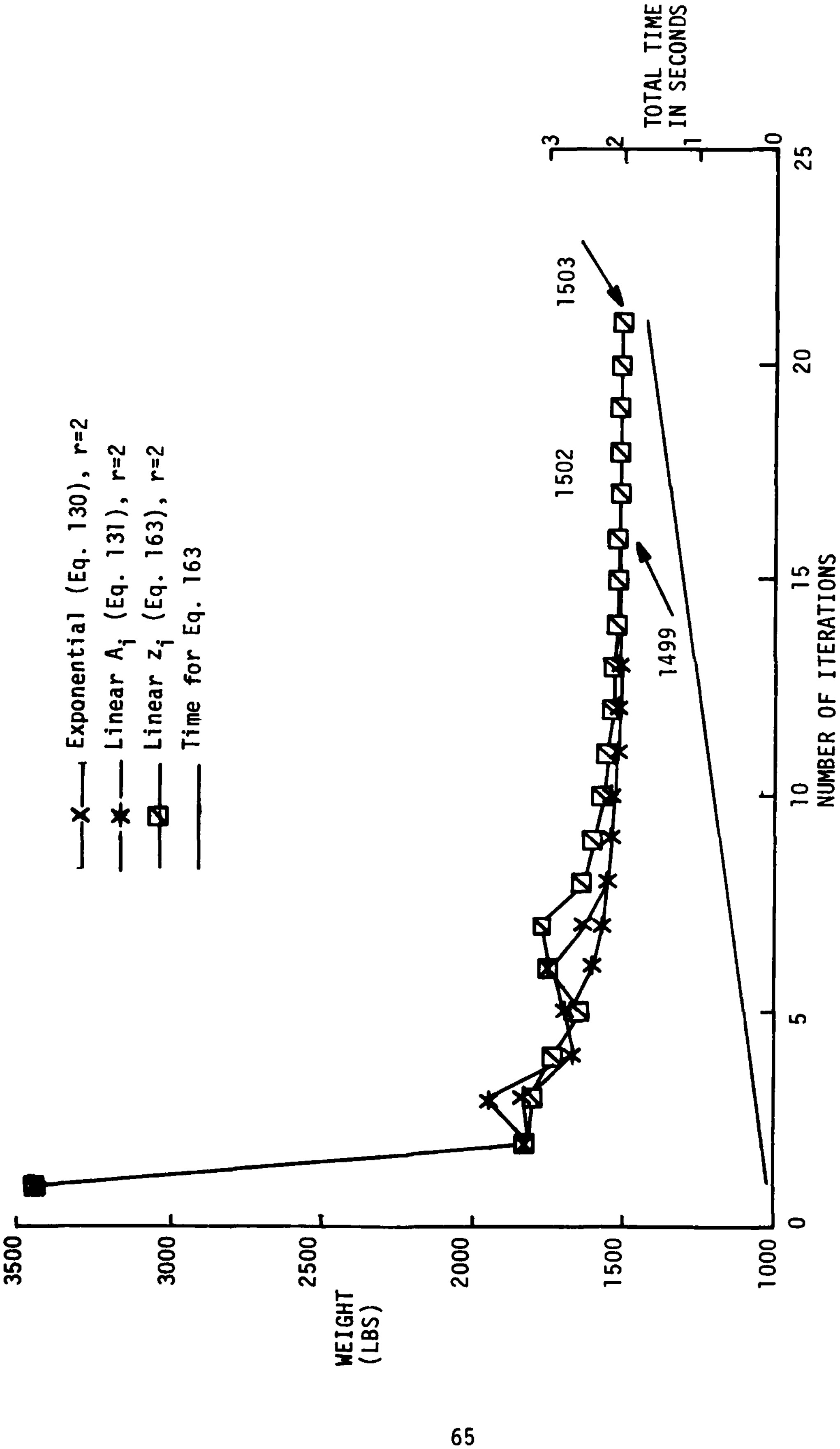


Figure 5. Iteration History for 10-Bar-Truss Stress Constraints - Case II

two approaches. In Case IIIa, the FSD algorithm (Eq. 117) was used. And in Case IIIb, the design variables were modified by using the exponential relation (Eq. 130) and the Lagrange multipliers were determined by using Eq. 140. This equation assumes that there is no coupling effect and that the elements are fully stressed. The iteration history for the two cases and the computer time for Case IIIb are given in Table 6. The FSD algorithm gives a fully stressed design which is nonoptimum. The use of Eq. 130 to determine the Lagrange multipliers for Case IIIb gave a design with a weight of 1502.8 lb which is fully stressed but closer to the real optimum weight of 1497.6 lb. If the allowable stress in element 9 was increased to 50 ksi, the use of Eq. 130 gives a design with a weight of 1621 lb, which is in between the optimum design (1497.6 lb) and the design (1725 lb) obtained by using the FSD algorithm. The iteration history for Case IIIa and IIIb is also given in Figure 6.

TABLE 6

ITERATION HISTORY FOR 10-BAR-TRUSS
- STRESS CONSTRAINTS - CASE III

Iteration No.	Case IIIa	Case IIIb	Total Time in Seconds
1	3434.9	3434.9	0.07
2	1815.5	1787.9	0.16
3	1726.2	1667.6	0.24
4	1648.1	1569.2	0.32
5	1590.6	1545.4	0.40
6	1570.4	1539.2	0.48
7	1568.5	1531.1	0.55
8		1523.9	0.63
9		1517.3	0.70
10		1512.6	0.78
11		1504.8	0.86
12		1502.0	0.94

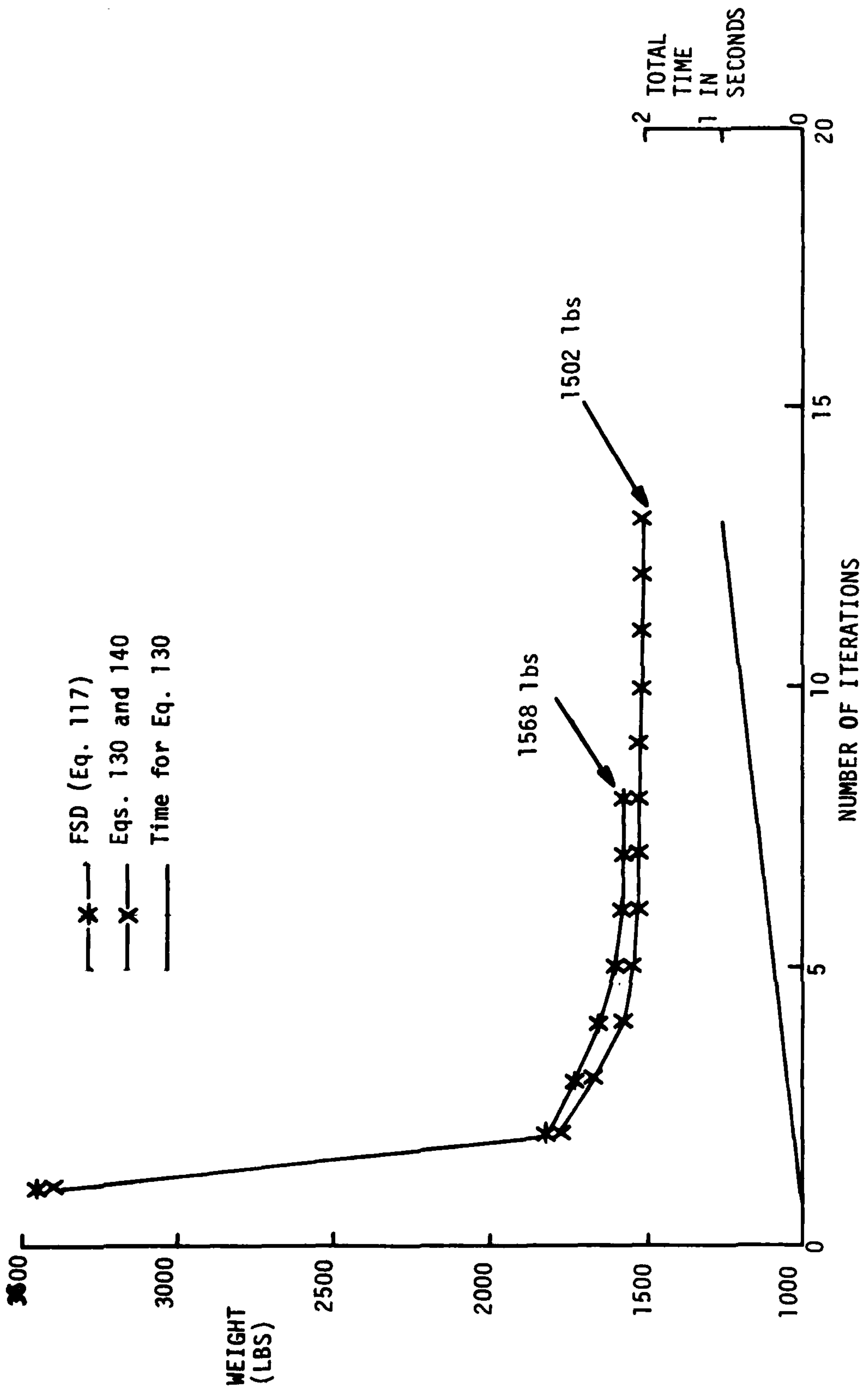


Figure 6. Iteration History for 10-Bar-Truss Stress Constraints - Case III

EXAMPLE 3. TWO-HUNDRED-BAR TRUSS (DISPLACEMENT AND STRESS CONSTRAINTS)

The 200-bar truss shown in Figure 7 is subjected to five loading conditions as follows:

(1) A load of 1000 lb in the positive X direction applied at nodes 1, 6, 15, 20, -----, 71.

(2) A load of 1000 lb in the negative X direction applied at nodes 5, 14, 19, -----, 75.

(3) A load of 1000 lb in the negative Y direction at nodes 1, 2, 3, 4, 5, 6, 8, 10 12, 14, 15, 16, -----, 73, 74, 75.

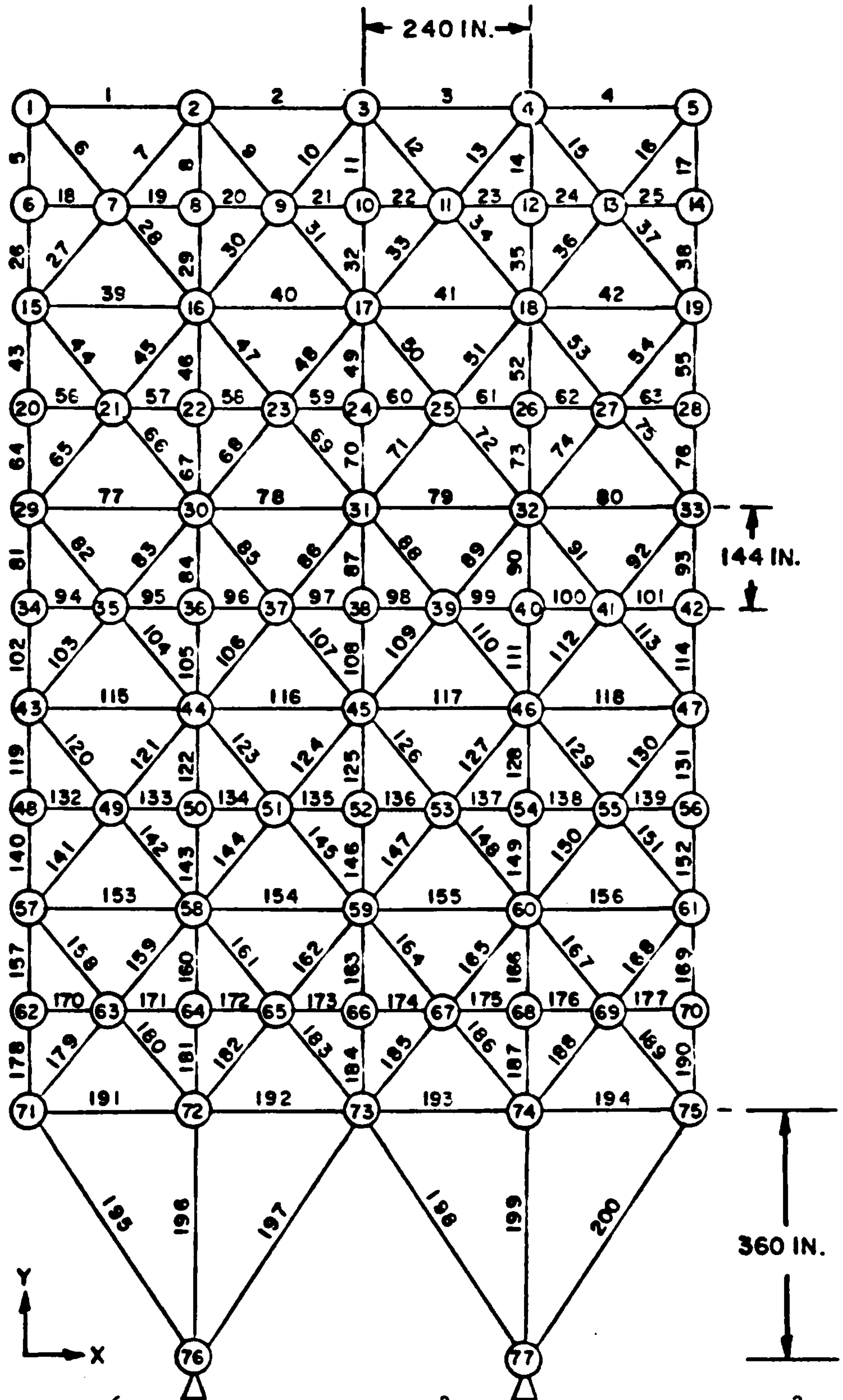
(4) Loading conditions 1 and 3.

(5) Loading conditions 2 and 3.

The maximum allowable stress in all the elements is equal to ± 10 ksi. The displacement limit in the X and Y directions at all the nodes is ± 0.5 in.

The structure was designed to satisfy stress and displacement constraints. Two approaches were used. In the first approach the constraints on the displacements and the stresses were both treated as potentially active constraints, i.e., the Lagrange multipliers associated with the stress as well as the displacement constraints were determined. In the second approach, the stress constraints were treated as passive constraints, i.e. no Lagrange multipliers or gradients of the constraints associated with stresses in the elements were evaluated. The Lagrange multipliers corresponding to only the displacement constraints were used in the recurrence relation. The linear recurrence relations for A_i (Eqs. 71 and 131) were used. The effect of using other recurrence relations may be found in Reference 31.

The step-size parameter 'r' was set equal to '2', and the Lagrange multipliers were determined by using the linear equations (Eqs. 86, 136). The structure was reanalyzed for both cases after each iteration. The symmetry of the structure was not taken into consideration in order to



$E = 30 X 10^6$ psi $\rho = 0.283 lbs/in^3$ Min. Size = $0.1 in^2$

Figure 7. Two-Hundred-Bar-Truss

reduce the loading conditions or the number of design variables. The number of potentially active constraints is determined by using the method discussed in Example 1. The iteration history for the two cases, Case I (stress + displacements) and Case II (displacements), is given in Table 7. The table also gives the CP time required for each iteration. The average CP time to analyze the structure was found to be 2.4 seconds. The difference between the time given in the table and 2.4 seconds is the time required to complete an optimization phase of the iteration. The optimization phase includes evaluating the constraint gradients, assembling the elements of Eqs. 86 or 136, solving the equations to determine the Lagrange multipliers, and modifying the design variables by using the recurrence relation. The iteration history for the two cases is also given in Figure 8.

The weight of the optimum design for this structure is 28858.4 lb. The areas of the elements associated with this design may be found in Reference 31.

EXAMPLE 4. CANTILEVER BOX-BEAM

The structure shown in Figure 9 was idealized with membrane quadrilateral elements, shear panels, and bar elements (posts). The quadrilateral membrane elements in the top and bottom skins consisted of four layers with fibers in the 0^o, 90^o, $+45^o$, and -45^o direction. The 0^o fibers are parallel to the length of the beam. The idealized structure consisted of 18 quadrilaterals, 18 shear panels and 18 posts. The six quadrilaterals near the tip of the box beam consisted of four layers of graphite epoxy. The six quadrilateral elements in the middle consisted of boron epoxy in the 0^o direction and the remaining three layers of graphite epoxy. The six quadrilateral elements closer to the fixed end consisted of all layers of boron-epoxy. The shear panels and posts have different material properties. The composite elements were designed by using the maximum stress criteria. The material properties used for this example are given in Table 8. The reason for selecting diverse properties for the different elements was to illustrate the versatility of the computer program. The program used to solve the problem is documented in Reference 33.

TABLE 7

ITERATION HISTORY FOR 200-BAR TRUSS

Iteration No.	Case I				Case II			
	Weight	Number of Active Constraints		Time per Iteration in Seconds	Weight	Number of Active Constraints		Time per Iteration in Seconds
		Disp.	Stress			Disp.	Stress	
1	144769	0	2	3.05	144769			
2	86154	0	4	3.49	42258*	4	-	3.44
3	60149	0	4	3.49	36277	3	-	3.69
4	47166	2	0	3.97	36038	3	-	5.75
5	36968	3	2	4.29	40273	2	-	3.95
6	33026	5	0	4.39	38949	3	-	6.13
7	31013	5	0	4.33	30436	5	-	5.86
8	30260	5	1	4.70	30976	7	-	7.35
9	29803	5	3	6.62	29285	9	-	10.90
10	29676	6	16	15.54	29579	-	-	-
11	29607	5	17	14.26				
12	29563	5	17	14.80				
20	29245	5	17	10.03				
30	29121	5	17	-				
40	29015	5	17	-				
50	28941	5	17	-				

* F.S.D. Design

Average Time For Analysis of the Structure - 2.4 seconds.

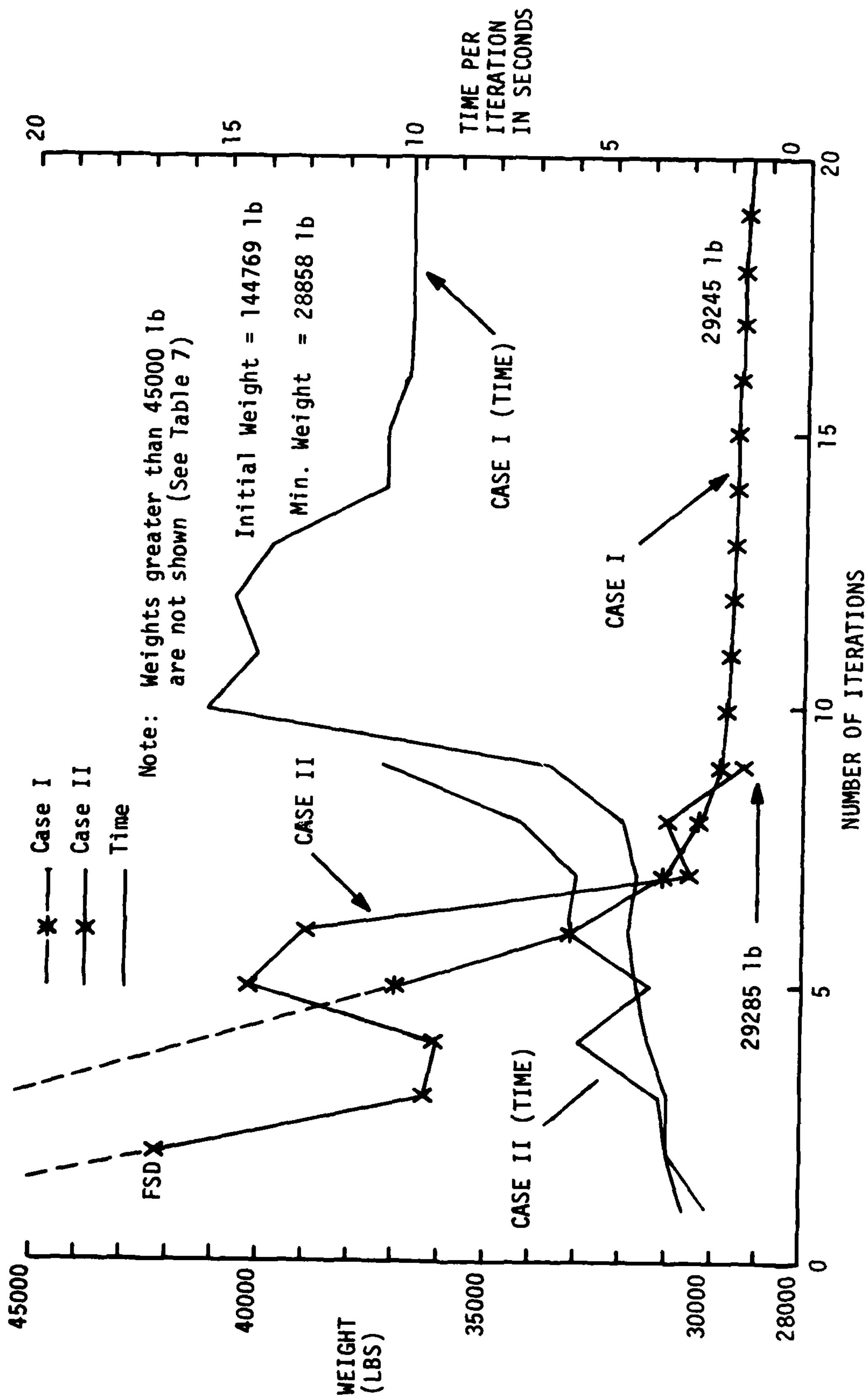


Figure 8. Iteration History for 200-Bar-Truss Stress and Displacement Constraints

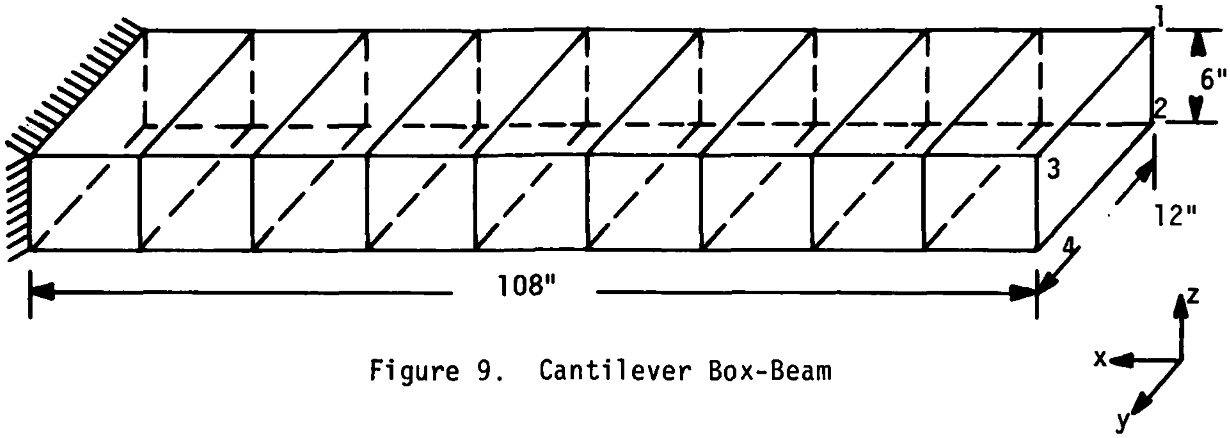


Figure 9. Cantilever Box-Beam

TABLE 8

MATERIAL PROPERTIES FOR EXAMPLE 4

	graphite-epoxy	boron-epoxy
E_{11}	18.5×10^6 psi	32.0×10^6 psi
E_{22}	1.6×10^6 psi	3.5×10^6 psi
ν_{12}	0.208	0.25
ν_{21}	0.0203	-
G	0.65×10^6 psi	0.93×10^6 psi
Density	0.055 lbs/in^3	o.0725 lbs/in^3
Thickness of laminae	0.0052	0.0052
Allowable Stress fiber direction		
tension	139.0 $kips/in^2$	166.0 $kips/in^2$
compression	92.4 $kips/in^2$	86.0 $kips/in^2$
Transverse direction		
tension	4.95 $kips/in^2$	6.0 $kips/in^2$
compression	29.7 $kips/in^2$	11.86 $kips/in^2$
Shear Stress	4.68 $kips/in^2$	3.95 $kips/in^2$

SHEAR PANELS

$E_1 = 30 \times 10^6$ psi, $\nu = 0.3$, Allowable Stress = 8.0 $kips/in^2$

$\rho = 0.28$ lbs/in^3

POSTS

$E_1 = 10.5 \times 10^6$ psi, $\nu = 0.3$, Allowable Stress = 25 $kips/in^2$

$\rho = 0.1$ lbs/in^3

The box was subjected to two loading conditions. In the first loading condition 1000 lb was applied at nodes 1 through 4 in the negative z direction. In the second loading condition, 500 lb was applied at nodes 2 and 4 in the negative y direction. The elements were designed by using the recurrence relation (Eq. 147) based on the strain energy stored in the element. To take into consideration the effect of the two loading conditions, the sum of the strain energies for both loading conditions was used. In the first iteration the number of layers in all four fiber directions for all the elements was assumed to be equal. The beam was first designed to satisfy the stress constraints. Then the design was modified to prevent local buckling of the elements. This was achieved by using the simple expression for the buckling of simply supported orthotropic plates (see Reference 33). The thickness of the plate elements was increased to satisfy the buckling equations, and the whole structure was reanalyzed to determine the modified in-plane forces. This process was continued until there was no further increase in the thickness of the plate elements. Since the buckling load of the plate can be further increased by increasing the thickness of the $\pm 45^{\circ}$ layers instead of 0° or 90° layers, only the thickness of these layers was primarily increased. The convergence behavior of the box-beam is given in Table 9. The minimum weight of a design satisfying only the stress constraints was 26.45 lb, and the design with stress and local buckling constraints was 34.40 lb. The distribution of the number of plies in the 0°, 90°, and $\pm 45^{\circ}$ direction is given in Figures 10 and 11. The number of plies in the $+45^{\circ}$ and -45° direction is equal. The number of plies in each layer was obtained by dividing the thickness of each layer by the ply thickness and rounding the result to the next higher integer. The number of plies in the top skin for both cases is nearly the same, but the number of plies in the bottom skin, which is subjected to compressive loads, has substantially changed. The change in the number of plies in the 0° and 90° direction is due to scaling the design variables and rounding the thickness to make it equal to a multiple of a single-ply thickness.

TABLE 9
ITERATION HISTORY FOR CANTILEVER BOX-BEAM

Iteration No.	Weight	Iteration No.	Weight
1	153.20	10	29.27
2	74.83	11	26.97
3	32.71	12	26.50
4	30.48	13	26.46
5	29.27	14	26.45
6	29.34	15	32.08*
7	29.62	16	33.90*
8	29.56	17	34.26*
9	29.50	18	34.36*
		19	34.40*

* - modified for local buckling

C.P. time for each iteration (1 - 14) 3.5 seconds

C.P. time for each iteration (15 - 18) 4.9 seconds

In the optimum design for the stress-constraint problem, note that the distribution of the number of plies in the top and bottom skin near the root is not the same. The distribution of layer thicknesses for the first 10 iterations, until the weight reached 29.27 lb, was symmetric; but with additional iterations the weight was reduced to 26.45 lb and the symmetry was lost. This may be due to unsymmetric loading conditions.

EXAMPLE 5. WING STRUCTURE

The finite element model of the wing structure (Reference 37) shown in Figure 12 was designed to satisfy (1) stress constraints, (2) negative twist constraints (a wash-out condition) and (3) positive twist constraints (a wash-in condition). The structure was subjected to a single loading condition. The coordinates of the node points and the loads applied at each node are given in Table 10. The structural model has 88 nodes and 158 members. The top and bottom skins were idealized by membrane quadrilaterals and triangles. The aluminum spars and ribs were idealized with shear panels and posts. The skin elements consist of four layers with fibers in 0^o, 90^o, $+45^o$, and -45^o directions. The 0^o fibers are

18	16	14	7	5	4	5	2	2
3	3	3	2	2	2	2	2	2
6	6	6	4	4	4	4	4	4

0°
90°
±45°

Top Skin

11	10	8	7	6	5	3	2	2
2	2	2	2	2	3	2	2	2
4	4	4	4	4	6	4	4	4

Bottom Skin

Figure 10. Optimum Design (Stress Constraints)

18	16	13	6	5	4	5	3	3
3	3	3	3	3	3	3	3	3
6	6	6	6	6	6	6	6	6

Top Skin

11	16	8	7	6	5	3	3	3
3	3	3	3	3	3	3	3	3
22	20	20	22	22	20	20	8	10

Bottom Skin

Figure 11. Optimum Design (Stress and Local Buckling Constraints)

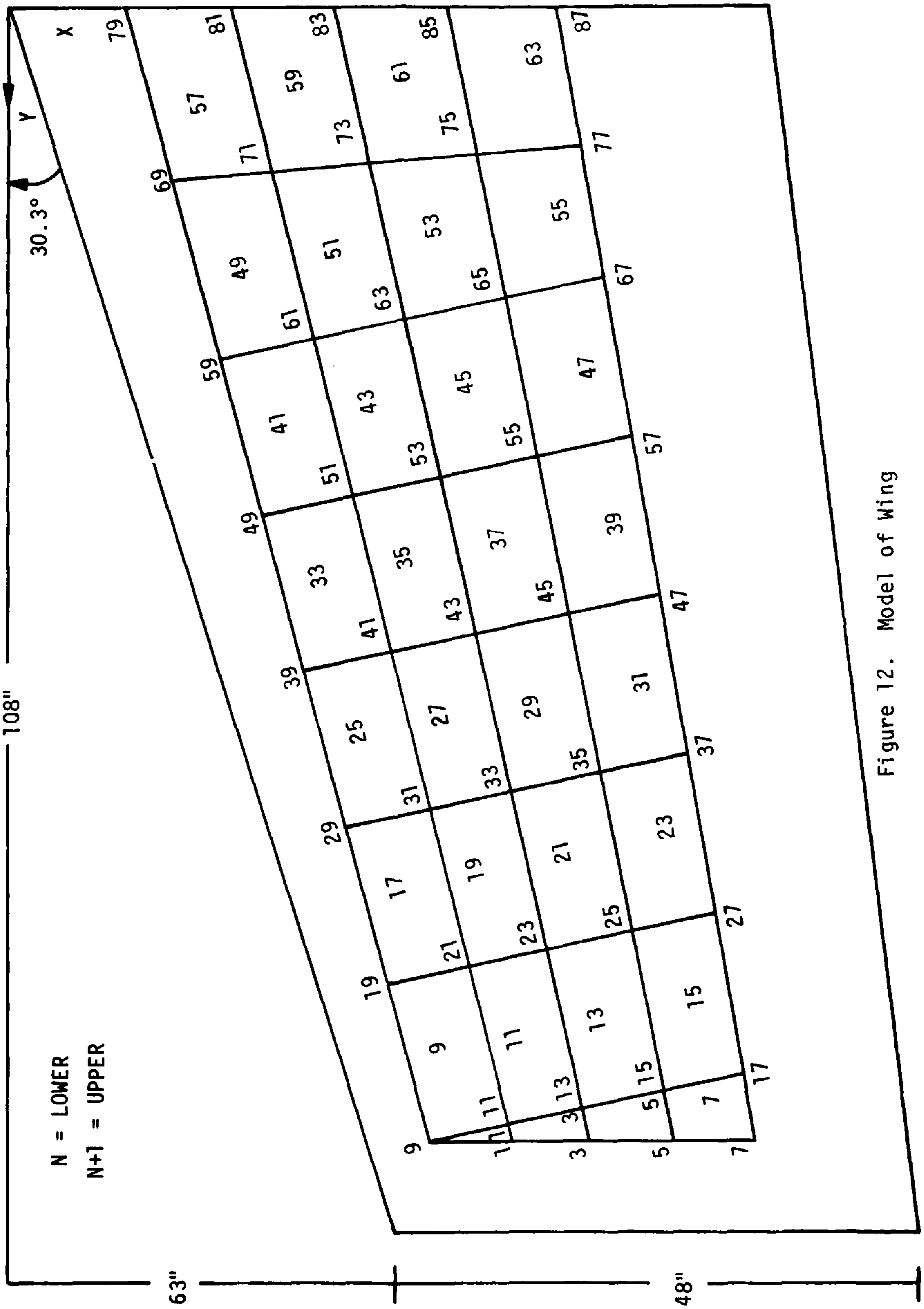


Figure 12. Model of Wing

TABLE 10

COORDINATES OF NODES AND APPLIED LOADS

JOINT	-X	-Y	-Z	FORCE-X	FORCE-Y	FORCE-Z
1	70.833	90.000	1.313	0.000	0.000	29.000
2	70.833	90.000	-1.313	0.000	0.000	29.000
3	78.167	90.000	1.500	-2800.000	-6960.000	1130.000
4	78.167	90.000	-1.500	2800.000	6960.000	1130.000
5	85.500	90.000	1.313	0.000	0.000	90.900
6	85.500	90.000	-1.313	0.000	0.000	90.900
7	92.833	90.000	1.125	-9870.000	-9780.000	1130.000
8	92.833	90.000	-1.125	9870.000	9780.000	1130.000
9	63.500	90.000	1.125	205.000	-7380.000	926.000
10	63.500	90.000	-1.125	-205.000	7380.000	926.000
11	69.686	87.471	1.349	0.000	0.000	178.000
12	69.686	87.471	-1.349	0.000	0.000	178.000
13	76.097	84.851	1.586	0.000	0.000	214.000
14	76.097	84.851	-1.586	0.000	0.000	214.000
15	82.745	82.133	1.427	0.000	0.000	253.000
16	82.745	82.133	-1.427	0.000	0.000	253.000
17	89.647	79.312	1.259	-5650.000	2320.000	1020.000
18	89.647	79.312	-1.259	5650.000	-2320.000	1020.000
19	57.265	77.669	1.279	2310.000	-946.000	723.000
20	57.265	77.669	-1.279	-2310.000	946.000	723.000
21	63.992	74.920	1.532	0.000	0.000	314.000
22	63.992	74.920	-1.532	0.000	0.000	314.000
23	70.952	72.071	1.799	0.000	0.000	326.000
24	70.952	72.071	-1.799	0.000	0.000	326.000
25	78.191	69.116	1.617	0.000	0.000	339.000
26	78.191	69.116	-1.617	0.000	0.000	339.000
27	85.692	66.050	1.424	-4070.000	1660.000	902.000
28	85.692	66.050	-1.424	4070.000	-1660.000	902.000
29	51.032	65.339	1.433	1740.000	-713.000	646.000
30	51.032	65.339	-1.433	-1740.000	713.000	646.000
31	58.297	62.369	1.715	0.000	0.000	340.000
32	58.297	62.369	-1.715	0.000	0.000	340.000
33	65.926	59.291	2.012	0.000	0.000	352.000
34	65.926	59.291	-2.012	0.000	0.000	352.000
35	73.635	56.100	1.807	0.000	0.000	365.000
36	73.635	56.100	-1.807	0.000	0.000	365.000
37	81.738	52.787	1.590	-4250.000	1740.000	974.000
38	81.738	52.787	-1.590	4250.000	-1740.000	974.000
39	44.799	53.008	1.587	1820.000	-743.000	694.000
40	44.799	53.008	-1.587	-1820.000	743.000	694.000
41	52.603	49.818	1.898	0.000	0.000	365.000
42	52.603	49.818	-1.898	0.000	0.000	365.000
43	60.691	46.512	2.225	0.000	0.000	378.000
44	60.691	46.512	-2.225	0.000	0.000	378.000
45	69.079	43.083	1.997	0.000	0.000	392.000
46	69.079	43.083	-1.997	0.000	0.000	392.000
47	77.784	39.525	1.756	-4440.000	1820.000	1050.000
48	77.784	39.525	-1.756	4440.000	-1820.000	1050.000
49	38.565	40.676	1.742	1890.000	-773.000	742.000

TABLE 10 (CONTINUED)

JOINT	-X	-Y	-Z	FORCE-X	FORCE-Y	FORCE-Z
50	38.565	40.678	-1.742	-1890.000	773.000	742.000
51	46.908	37.267	2.082	0.000	0.000	390.000
52	46.908	37.267	-2.082	0.000	0.000	390.000
53	55.555	33.732	2.438	0.000	0.000	404.000
54	55.555	33.732	-2.438	0.000	0.000	404.000
55	64.523	30.067	2.187	0.000	0.000	420.000
56	64.523	30.067	-2.187	0.000	0.000	420.000
57	73.830	26.262	1.922	-4640.000	1900.000	1120.000
58	73.830	26.262	-1.922	4640.000	-1900.000	1120.000
59	32.331	28.347	1.896	2290.000	-937.000	883.000
60	32.331	28.347	-1.896	-2290.000	937.000	883.000
61	41.214	24.716	2.265	0.000	0.000	413.000
62	41.214	24.716	-2.265	0.000	0.000	413.000
63	50.420	20.953	2.651	0.000	0.000	391.000
64	50.420	20.953	-2.651	0.000	0.000	391.000
65	59.967	17.050	2.376	0.000	0.000	369.000
66	59.967	17.050	-2.376	0.000	0.000	369.000
67	69.876	13.000	2.088	-3030.000	1240.000	804.000
68	69.876	13.000	-2.088	3030.000	-1240.000	804.000
69	25.166	14.173	2.073	3370.000	-520.000	1040.000
70	25.166	14.173	-2.073	-3370.000	520.000	1040.000
71	35.583	12.304	2.446	0.000	0.000	433.000
72	35.583	12.304	-2.446	0.000	0.000	433.000
73	46.181	10.403	2.827	0.000	0.000	370.000
74	46.181	10.403	-2.827	0.000	0.000	370.000
75	56.964	8.469	2.502	0.000	0.000	304.000
76	56.964	8.469	-2.502	0.000	0.000	304.000
77	67.938	6.500	2.169	-1370.000	262.000	446.000
78	67.938	6.500	-2.169	1370.000	-262.000	446.000
79	18.000	0.000	2.250	1730.000	0.000	544.000
80	18.000	0.000	-2.250	-1730.000	0.000	544.000
81	30.000	0.000	2.625	0.000	0.000	224.000
82	30.000	0.000	-2.625	0.000	0.000	224.000
83	42.000	0.000	3.000	0.000	0.000	190.000
84	42.000	0.000	-3.000	0.000	0.000	190.000
85	54.000	0.000	2.625	0.000	0.000	155.000
86	54.000	0.000	-2.625	0.000	0.000	155.000
87	66.000	0.000	2.250	-715.000	0.000	226.000
88	66.000	0.000	-2.250	715.000	0.000	226.000

parallel to the direction of the middle spar as defined by connecting nodes 4 and 84 in the top skin or 3 and 83 in the bottom skin. The elastic constants and the allowable strengths for graphite-epoxy and aluminum are given in Table 11. The maximum stress criteria was used to design the composite elements. For the first iteration for all three cases, the relative sizes of the bar elements and the thicknesses of the plate element were 1.0 to 0.1, respectively. The percentage of plies in the four fiber directions for the initial design were assumed to be equal.

(a) Stress-Constraint Design

The structure was designed by using the recurrence relation (Eq. 147) based on the strain energy in each layer and in each element. The initial weight of the structure, with an equal percentage of plies in the four fiber directions, was 312.84 lb, and the minimum-weight design obtained after 14 iterations was 45.45 lb. The iteration history is given in Table 12. The weight of the structure after 7 iterations was found to be 45.63 lb. The design was scaled after each iteration to obtain a feasible design. The distribution of the number of plies in the four fiber directions is shown in Figure 13. The tip deflections at nodes 7 and 9 are given in Table 13. The tip twists through an angle - 5.458^{o}. This is the angle between the lines joining nodes 9 and 7 before and after loading.

(b) Twist-Constraint Designs

The wing structure was designed to satisfy two twist constraints to illustrate the use of the algorithm. The two angles selected for this are -7^{o} (wash-out) and $+2^{o}$ (wash-in) (see Figure 14). The structure was designed by using Eq. 48 for a single-displacement constraint where the virtual load consisted of a unit couple. The stress constraints in the elements were treated as passive constraints. The iteration history for the wash-out and wash-in cases is shown in Figures 15 and 16, respectively. After each iteration the structure was scaled to satisfy the stress and twist constraints. The two designs in Figures 15 and 16 are referred to as the 'stress design' and the 'twist design'.

TABLE 11

PROPERTIES AND ALLOWABLE STRESSES FOR WING STRUCTURE

Properties	Graphite Epoxy	Aluminum
E_{11} (psi)	18.5×10^6	10.5×10^6
E_{22} (psi)	1.6×10^6	10.5×10^6
ν_{12}	0.25	0.3
Shear Modulus	0.65×10^6	4.038×10^6
Density (lbs/in^3)	0.055	0.1
Lamina Thickness (in)	0.0052	
Allowable Stresses (ksi)		
F_x (tension)	139.0	45
F_x (compression)	86.0	45
F_y (tension)	-	45
F_y (compression)	-	45
F_{xy}	46.8	25.9

TABLE 12

ITERATION HISTORY FOR WING STRUCTURE STRESS CONSTRAINTS

Iteration No.	Weight	Iteration No.	Weight
1	312.84	8	45.75
2	112.18	9	45.71
3	68.37	10	45.59
4	59.89	11	45.49
5	55.78	12	45.49
6	55.78	13	45.51
7	45.63	14	45.45

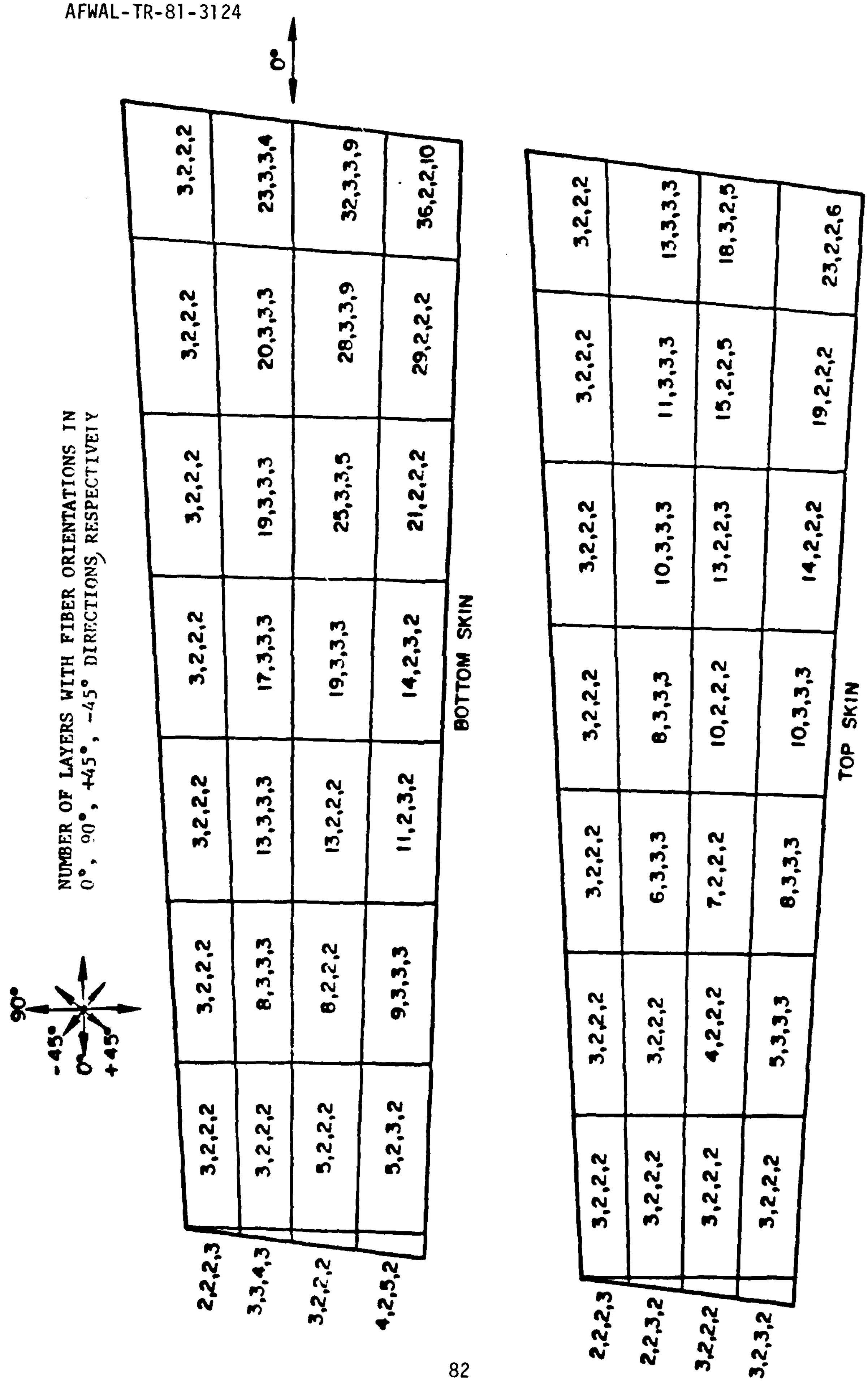


Figure 13. Optimum Design of Wing (Stress Constraint)

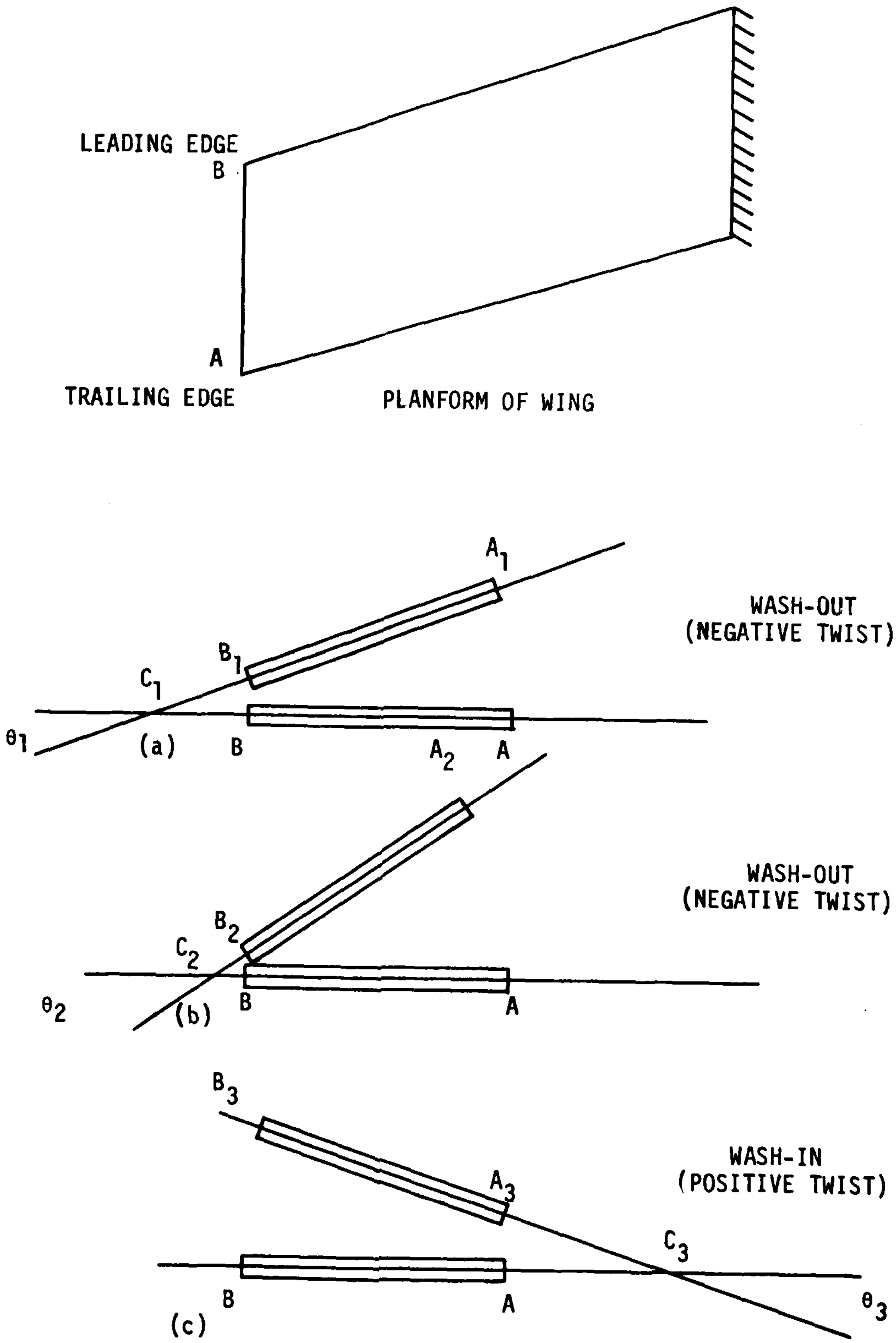


Figure 14. Definition of Twist

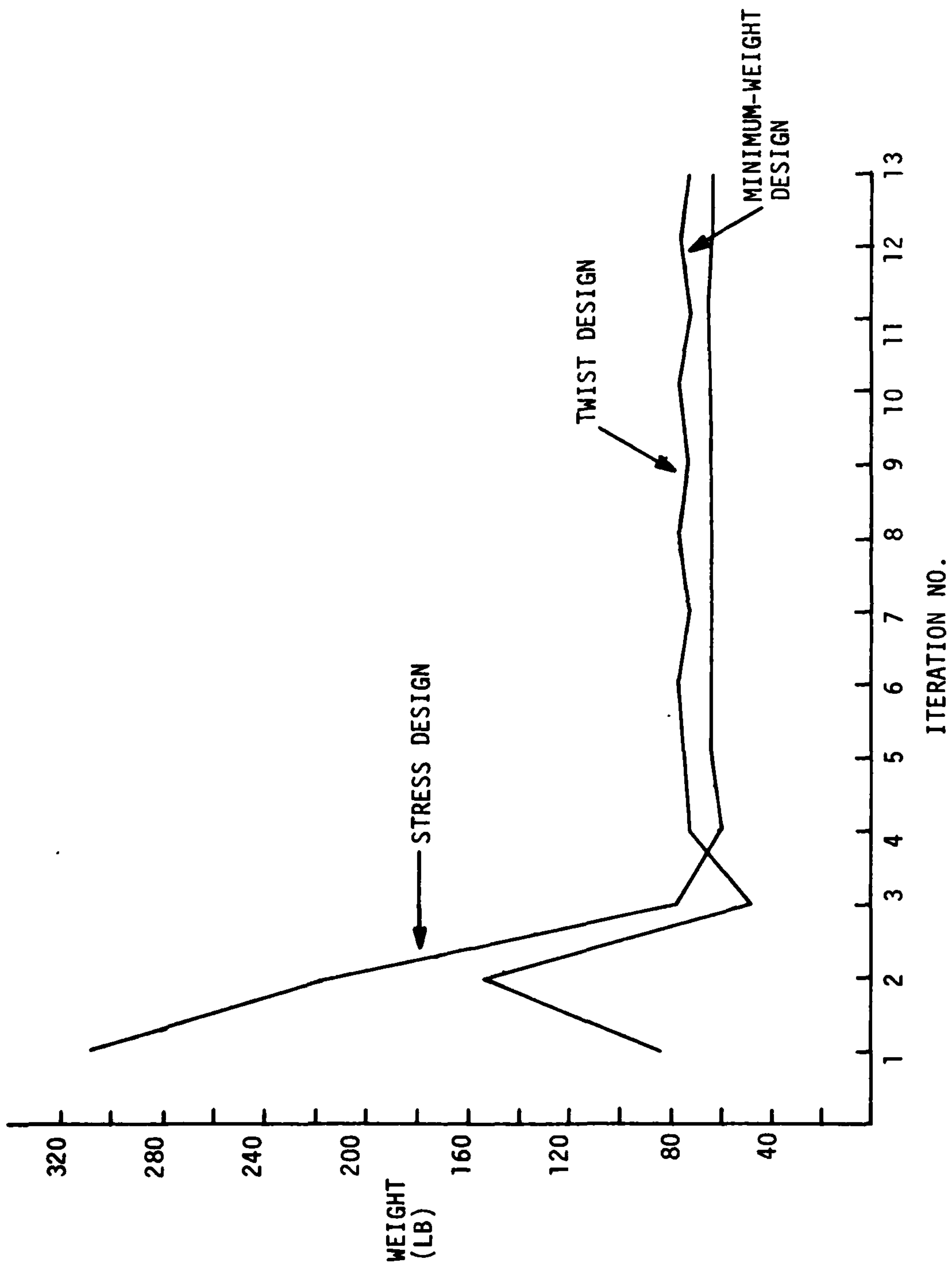


Figure 15. Iteration for Twist Constraints (Wash-out -7°)

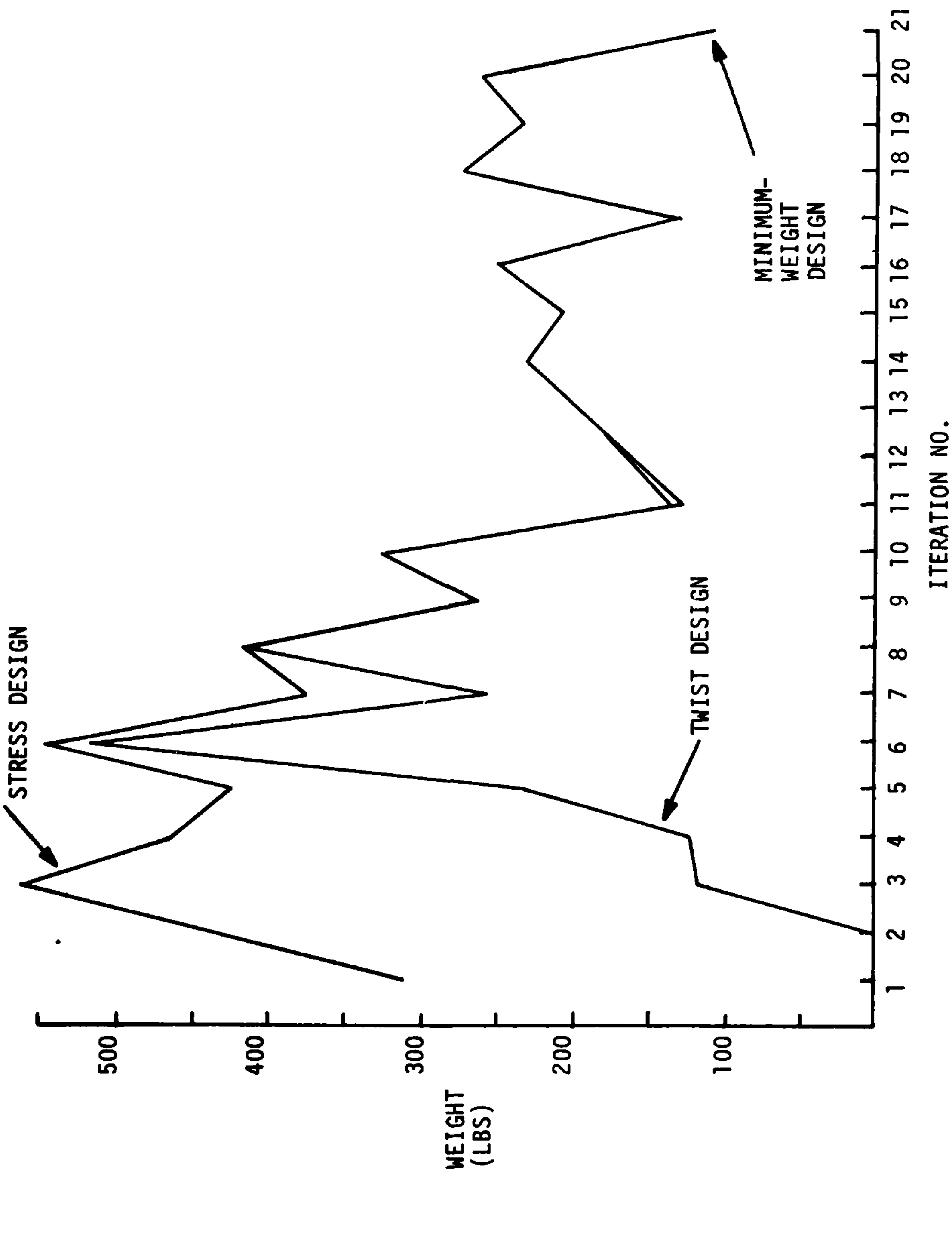


Figure 16. Iteration for Twist Constraint (Wash-in 2°)

In the 'stress design' the stresses in all the elements are less than orequal to the maximum allowable stress, and in the 'twist design' the twist of the tip of the wing is equal to the prescribed value.

For the wash-out case the lowest design weight of 71.86 lb was obtained in the 11th iteration. The weight after the 4th iteration, 72.22 lb, did not improve significantly with additional iterations. The displacements at nodes 7 and 9 for the least-weight design are given in Table 14, and the distribution of the plies in each element is given in Figure 17.

In the wash-in case the twist at the tip for the initial design was negative. This design could not be scaled to achieve the $+2^{\circ}$ twist. From the 2nd to the 8th iteration the twist of the stress design increased until it reached $+2^{\circ}$. In the subsequent iterations for all cases, the twist designs were acceptable since they satisfied the stress constraints also. Figure 16 shows that the weight of the structure oscillates and does not reach a stable position in the design space. The least-weight design, with a weight of 105.54 lbs was obtained at the 21st iteration. The oscillations may be due to the large value of the step-size parameter, r=2, selected for this problem.

The displacement at nodes 7 and 9 for this design are given in Table 15. Figure 18 shows the distribution of the plies in each element. The figure shows a heavy concentration of 0° plies only on one side of the wing. A design with an even distribution of plies would have resulted for both cases if the 0° plies were placed at an angle instead of parallel to the line joining nodes 3 and 83.

EXAMPLE 6. DOME STRUCTURE

The dome structure shown in Figure 19 was optimized to satisfy system stability. The structure was subjected to a load of 1000 lb applied in the vertical direction at each node point. The structure has 61 nodes and 132 members. The dome is supported at all the node points on the boundary and all the joints are hinged. The structure was idealized with bar elements carrying axial load only.

TABLE 13

DEFLECTIONS AT TIP FOR THE STRESS-CONSTRAINT DESIGN

Node 7 (trailing edge) Direction (in)			Node 9 (leading edge) Direction (in.)			Twist (deg.)
X	Y	Z	X	Y	Z	
-0.06	-.19	13.43	-.04	-.16	10.62.	-5.45

TABLE 14

DEFLECTIONS AT TIP FOR THE
TWIST-CONSTRAINT DESIGN
(WASH-OUT 7°)

Node 7 (trailing edge) Direction (in.)			Node 9 (leading edge) Direction (in)			Twist (deg.)
X	Y	Z	X	Y	Z	
-.09	-.15	11.68	-.08	-.10	8.07	-7.00

TABLE 15

DEFLECTIONS AT TIP FOR THE
TWIST-CONSTRAINT DESIGN
(WASH-IN 2°)

Node 7 (trailing edge) Direction (in)			Node 9 (leading edge) Direction (in.)			Twist (deg.)
X	Y	Z	X	Y	Z	
.06	-.10	4.49	.04	-.11	5.51	2.00

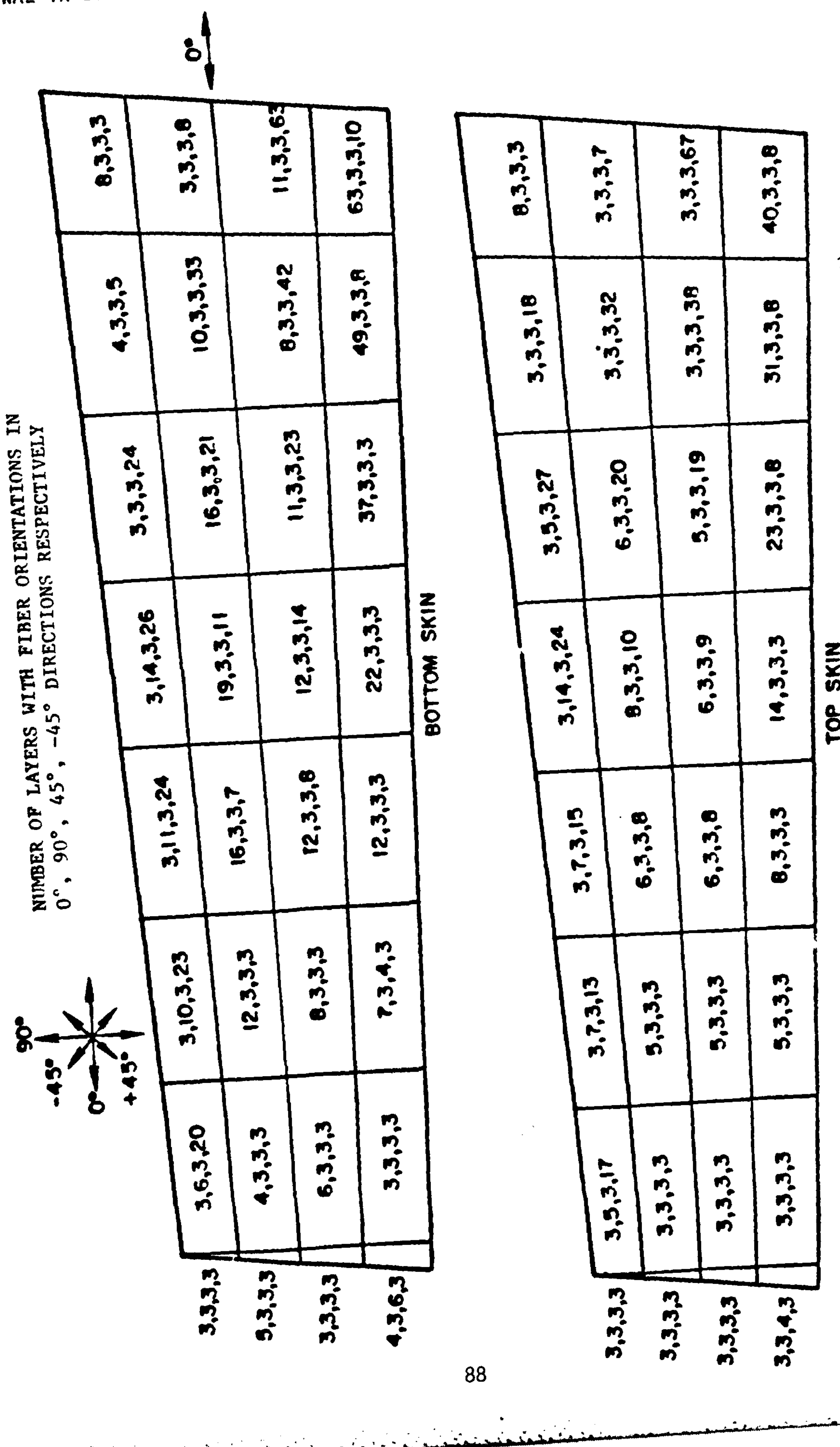


Figure 17. Optimum Design of Wing (Twist Constraint, Wash-out -7°)

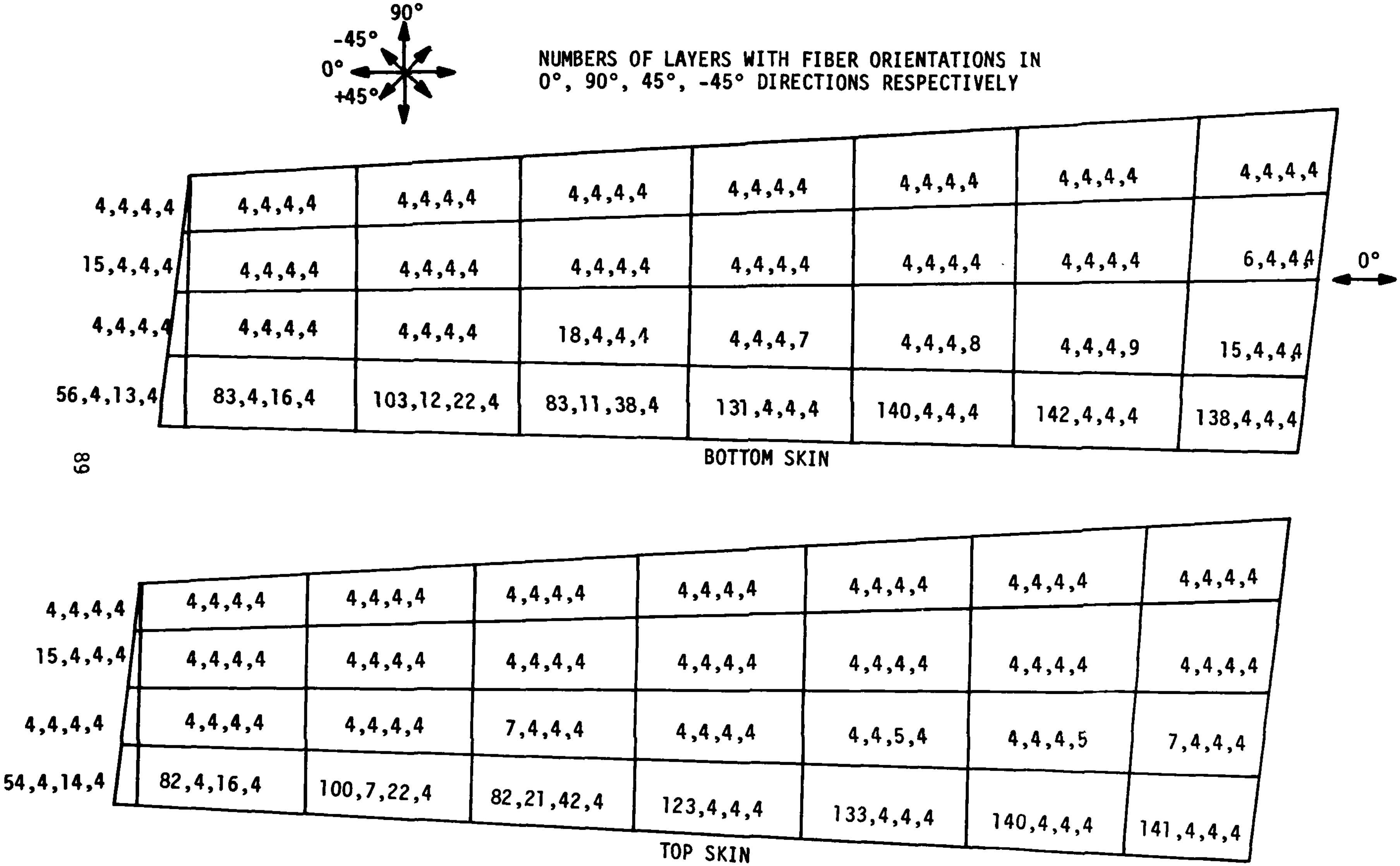


Figure 18. Optimum Design of Wing (Twist Constraint, Wash-in 2°)

$E = 10^7$ psi

Stress Limit = 75,000psi

Special wt = 0.1

Minimum Size = 0.02 (Relative)

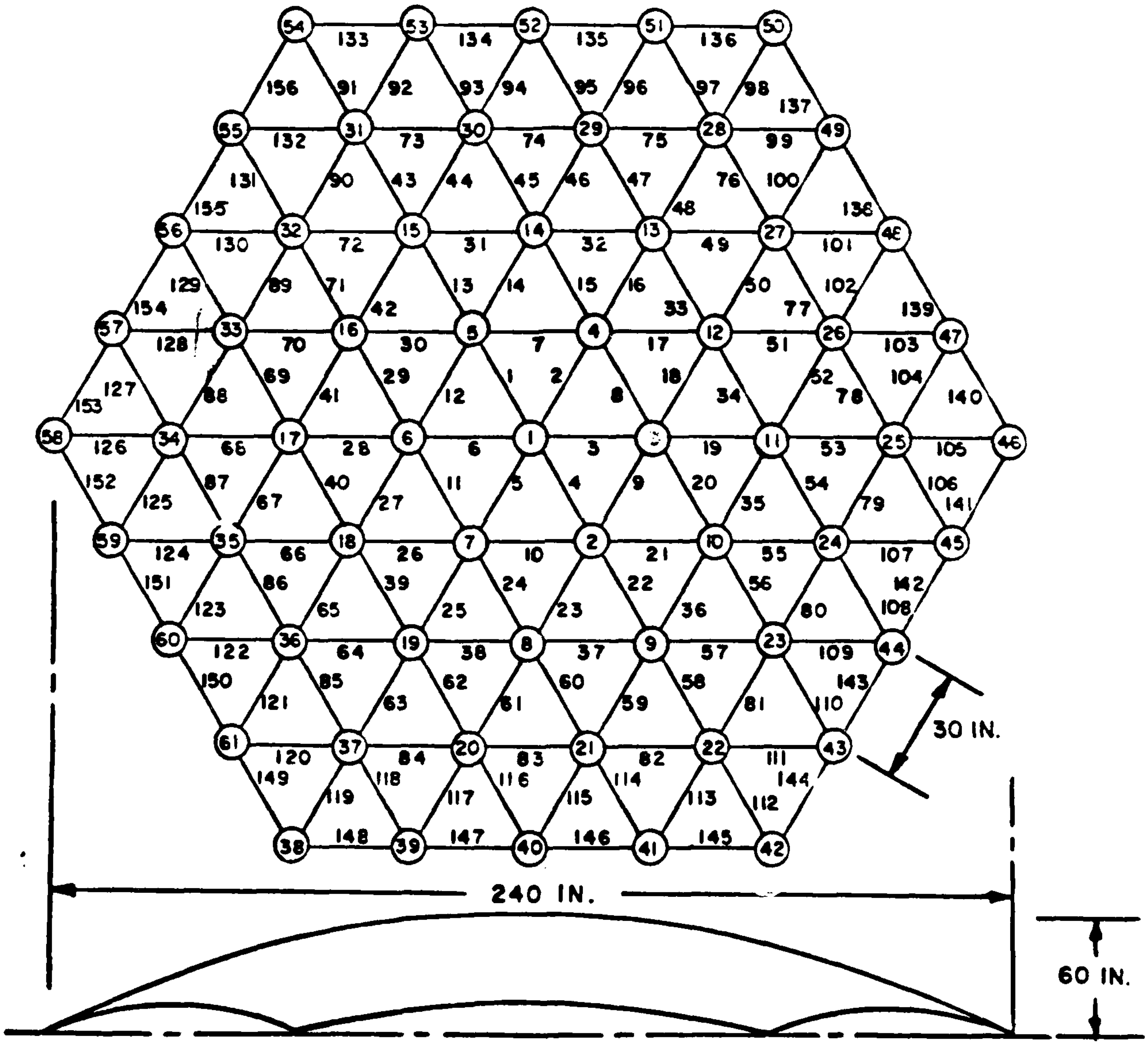


Figure 19. Geodesic Dome

The structure was designed by using the recurrence relation (Eq. 175) which uses the strain energy in an element in the buckled mode associated with the lowest buckling load of the structure. The stress constraints in the elements were treated as passive constraints. The structure was designed by using two values of the step-size parameter, r=2 (Design I) and r=10 (Design II). The iteration history for the two cases is given in Tables 16 and 17, respectively. The minimum-weight designs are given in Table 18.

In these tables 'Design B' is the scaled design satisfying the stress constraints in the elements. The buckling load of 'Design E' is equal to the applied load. For all the designs the system buckling is the active constraint. The stress constraint, because of the high allowable, does not become active.

The step-size parameter r=2 was too large to obtain good convergence. Therefore the step-size was reduced by generating the intermediate design vector whenever the scaled weight of the structure after each iteration was greater than the previous lower-weight design. (See discussion in Section VIII). The intermediate designs obtained by this procedure are indicated by '**' in Table 16. In the case of the step-size parameter r=10, the minimum weight design was obtained after three iterations. The weights of the optimum designs obtained for the two step-size parameters are nearly equal. However, Table 18 shows that the two designs are not identical.

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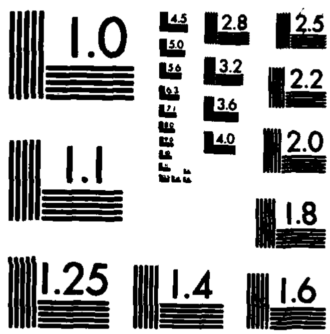

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TABLE 16

ITERATION HISTORY OF DOME STRUCTURE
FOR m = 2 AND INTERMEDIATE DESIGN
VECTORS (DESIGN I)

	Design B		Design E	
Cycle No.	Wt (lbs)	Buckling Load	B.Load / Wt.	Wt (lbs)
1	16.583	0.1877	0.0113	87.836
2	91.767	0.6073	0.0066	151.086
3	19.503	0.2679	0.0137	72.774**
4	45.307	0.4551	0.0100	99.537
5	25.144	0.3084	0.0122	81.511
6	21.708	0.3198	0.0147	67.859

*Load in kips over each node point

**Weight corresponding to intermediate design vectors

Initial Design - Area of all members = 0.2160 in^2

TABLE 17

ITERATION HISTORY OF DOME STRUCTURE
FOR m = 10 (DESIGN II)

	Design B		Design E	
Cycle No.	Wt (lbs)	Buckling* Load	B.Load / Wt.	Wt (lbs)
1	16.583	0.1887	0.0113	87.836
2	21.157	0.2949	0.0139	71.729
3	24.512	0.3655	0.0149	67.059

*Load in kips over each node point

Initial Design - Area of all members = 0.2160 in^2

TABLE 18

OPTIMUM DESIGN OF GEODESTIC DOME

Elements	Design I Area (in^2)	Design II Area (in^2)
1,2	0.2016	0.2136
7	0.2805	0.2648
13,16	0.2566	0.2469
14,15	0.1775	0.1863
31,32	0.1819	0.1936
43,48	0.1391	0.1378
44,47	0.2095	0.2134
45,46	0.1760	0.1847
73,75	0.1245	0.1115
74	0.1753	0.1837
91,98	0.1261	0.1125
92,97	0.1355	0.1336
93,96	0.1253	0.1119
94,95	0.1231	0.1079

Because of the symmetry only the areas of elements within one sector are given.

SECTION X

SUMMARY AND CONCLUSIONS

The minimum-weight design must satisfy the optimality criterion derived for the specific type of constraints imposed on the structure. The criterion states that a quantity $\psi = \sum_{j=1}^{m} \lambda_j \frac{F_{ij}}{\rho_i \ell_i A_i^2}$ associated with all the elements in the structure is equal to unity. The quantity F_{ij} depends on the nature of the constraints. In the case of displacement, stress and stability constraints F_{ij} is equal to Q_{ij}, R_{ij} and $\overline{Q}_{ij}$ respectively and is defined in Eqs. 10, 125 and 171. F_{ij} is a function of the energy stored in the element, not the nature of the energy depends upon the type of constraints.

The algorithms based on an optimality criterion are iterative and use the recurrence relation derived from the optimality criterion to modify the design variables. There are basically three distinct relations which we have derived from the optimality criterion. These are (1) the exponential relation, (2) the linear relation for A_i, and (3) the linear relation for z_j. Many other recurrence relations can be written from these thre by selecting different values of the step-size parameter r. Each of the recurrence relations contains the quantity ψ. Use of the recurrence relation moves the initial design to the optimum design satisfying the optimality criterion. When the condition ψ=1 is satisfied, the recurrence relation does not change the design variables. However, in the optimization algorithm based on the optimality criterion some of the constraints are treated as passive and the step-size parameter is generally kept constant; therefore the scaled weight of the structure reaches a certain minimum value, and then it increases. The real minimum weight lies closer to the lowest-weight design obtained by using the algorithm. The real minimum-weight design satisfying the optimality criterion can be obtained from the lowest weight with additional iterations but with a smaller step size. The difference between the real minimum and the lowest-weight design is generally small. This behavior is particularly true when the stress dominate.

The recurrence relations can only be used after determining the coefficients F_{ij} and the Lagrange multipliers. The coefficients F_{ij} are related to the gradients of the constraints. The number of Lagrange multipliers depends upon the number of active constraints. The active constraints are defined as those that are associated with the positive Lagrange multipliers. These constraints are very close to the constraint surface. The number of active constraints changes with the iterations. Initially the number is small, but as the design approaches the optimum the number increases. The constraints sometimes switch from the active category to the passive category, depending upon their activity as the optimum process proceeds from one iteration to the next. A prior knowledge of the active constraints at the optimum is not necessary. At each iteration one has to determine the set of active constraints.

Since we have not presented any results using the Newton-Raphson method to determine the Lagrange multipliers, we will not discuss the advantages or disadvantages of this approach. However, in using this approach the coefficients Q_{ij} need to be evaluated for all the elements and constraints, and the initial values of the Lagrange multipliers need be assumed. Also in the Newton-Raphson approach it is generally difficult to eliminate the equations associated with the passive constraints.

The last approach, based on using simple relations to determine the Lagrange multipliers, is computationally efficient. It requires less CP time to perform the optimization phase of the algorithm. With this approach it is possible to combine all the virtual loads associated with all the constraints into one equivalent virtual load, since the Lagrange multipliers can be estimated before evaluating the coefficients Q_{ij} or R_{ij}. Particularly for the stress-constrained problem, if the initial values of the Lagrange multipliers are assumed to be proportional to the forces in the bars and the exponential recurrence relation is used to modify the design variables, use of Eq. 132 leads to the near-optimum-weight design with the least computational effort. However, if the constraints imposed on the structure are mixed, such as displacements and stresses, this algorithm needs a large number of iterations and the scaled weight of the structure may violently oscillate. The approximate relation (Eq. 140) to determine the Lagrange multiplier for the FSD design gives

a near-minimum-weight design if the minimum-weight design is a FSD. The FSD algorithm (Eq. 116) does not lead to the correct minimum-weight design if there are large differences in the maximum allowable stresses in the elements.

In the optimization of a structure one may desire to obtain a design that satisfies the theoretical optimality criterion. However, in a practical design problem, the true optimum may not always be feasible owing to manufacturing considerations. In many practical cases it may be useful to obtain a design that is a near-minimum-weight design with a good distribution of material rather than a design satisfying the theoretical optimality criterion. For many problems such a design can be obtained by including only a small number of constraints into the potentially active category and treating the other constraints as passive constraints. A good example of this is the displacement-stress-constraint problem. For such a problem it is advantageous to treat all the stress constraints as passive constraints instead of including them in the constraint equations and evaluating their gradients and associated Lagrange multipliers. Since the number of active stress constraints is large compared to the displacement constraints, this simplification saves substantial CP time. This and other simple relations proposed in the previous sections to determine the Lagrange multiplier minimizes the total computational effort needed to design an optimum structure.

APPENDIX

THREE-BAR TRUSS PROBLEM

A three-bar truss shown in Figure A-1 with the variation of loads and constraints will be used to illustrate the application of the optimality criterion approach to design a minimum-weight structure. The problem is small and most of the required calculations can be done by using a calculator.

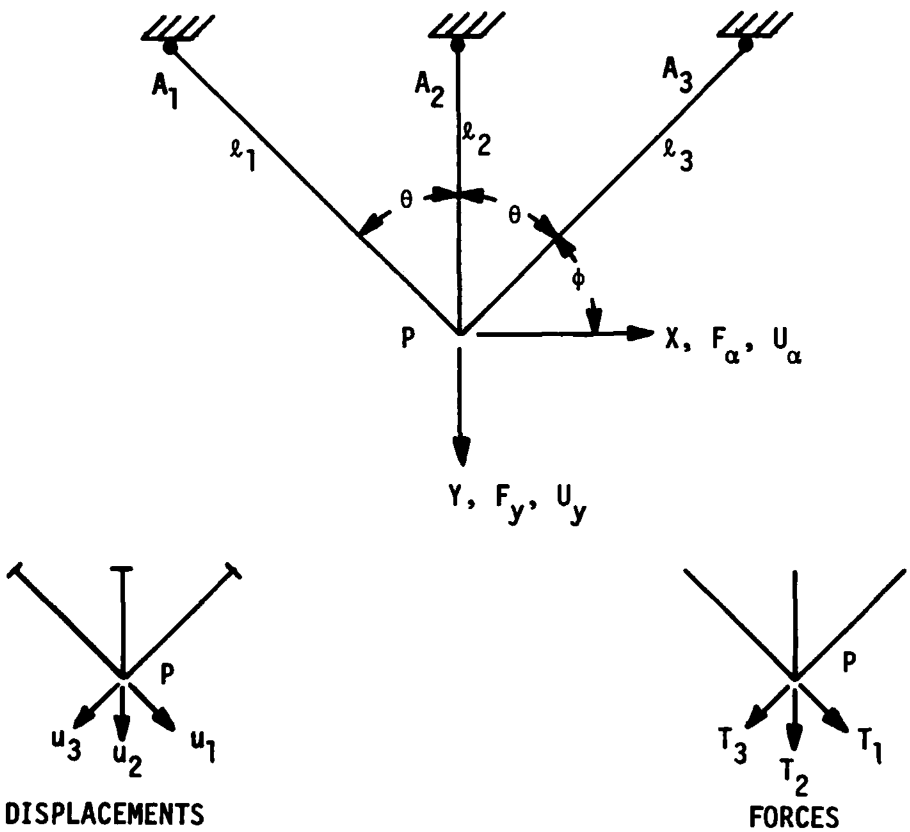


Figure A-1. Three-Bar Truss

Take horizontal and vertical displacements of the node P are denoted by U_x and U_y. The forces applied at the node in the X and Y directions are F_x and F_y, respectively. The displacements U_1, and U_2 and U_3 of the point P in the directions parallel to the length of the three bars are given by

$$\begin{aligned} U_1 &= U_x \cos\phi + U_y \cos\theta \\ U_2 &= U_y \\ U_3 &= -U_2 \cos\phi + U_y \cos\theta \end{aligned} \tag{A-1}$$

The bar forces are given by

$$T_1 = \frac{A_1 E U_1}{\ell_1} \qquad T_2 = \frac{A_2 E U_2}{\ell_2} \qquad T_3 = \frac{A_3 E U_3}{\ell_3} \tag{A-2}$$

where A_1, A_2, and A_3 are the cross-sectional areas of the three elements, and ℓ_1, ℓ_2, and ℓ_3 are their respective lengths.

The stresses in the elements can be written as

$$\sigma_1 = \frac{T_1}{A_1} \qquad \sigma_2 = \frac{T_2}{A_2} \qquad \sigma_3 = \frac{T_3}{A_3} \tag{A-3}$$

The equilibrium equations can be written by taking a summation of the forces at node P. This gives

$$\begin{aligned} T_1 \cos\phi - T_3 \cos\phi &= F_x \\ T_1 \cos\theta + T_2 + T_3 \cos\theta &= F_y \end{aligned} \tag{A-4}$$

Using Equations A-1 and A-2, these equations can be written as

$$\begin{bmatrix} K_1 & K_1 \\ K_2 & K_2 \end{bmatrix} \begin{Bmatrix} U_x \\ U_y \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \tag{A-5}$$

where

$$\begin{aligned} K_1 &= \left(\frac{A_1}{\ell_1} + \frac{A_3}{\ell_3} \right) E \cos^2\phi \\ K_1 &= \left(\frac{A_1}{\ell_1} - \frac{A_3}{\ell_3} \right) E \cos\phi \cos\theta \\ K_3 &= \left(\frac{A_1}{\ell_1} + \frac{A_3}{\ell_3} \right) E \cos^2\theta + \frac{A_2}{\ell_2} E \end{aligned} \tag{A-6}$$

Equations A-5 are the load-displacement relations and are equivalent to Equation 1 in Section I. Solving Equation A-5, the nodal displacements are given by

$$\begin{aligned} U_x &= \frac{K_2 F_y - K_3 F_x}{K_2^2 - K_1 K_3} \\ U_y &= \frac{K_2 F_\alpha - K_1 F_y}{K_2^2 - K_1 K_3} \end{aligned} \tag{A-7}$$

The displacements U_x and U_y are also be given by

$$U_x = \sum_{i=1}^{3} \frac{T_i\, t_i^1\, \ell_i}{A_i E} \quad ; \quad U_y = \sum_{i=1}^{3} \frac{T_i\, t_i^2\, \ell_i}{A_i E} \tag{A-8}$$

where t_i^1 and t_i^2 are the forces in the ith bar, due to a virtual load of unit magnitude, applied at node P in the x and y directions, respectively.

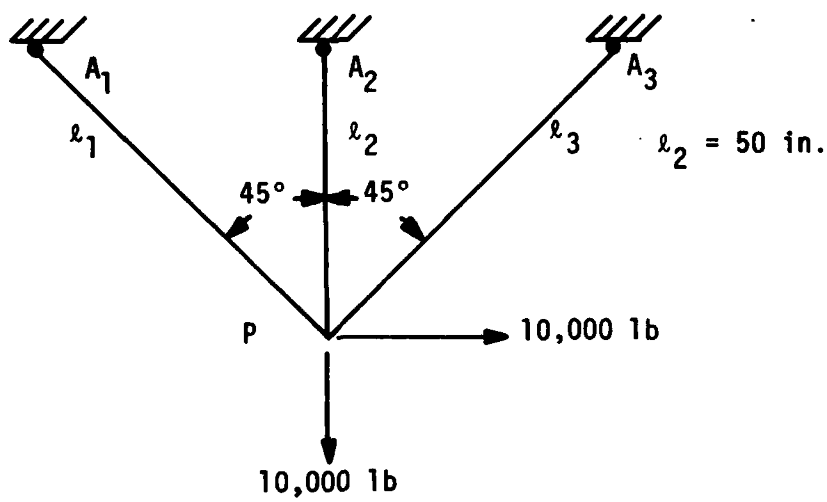
The equilibrium equations for the virtual load system $\{f_x, f_y\}$ can be written by replacing the vector $\{F_x, F_y\}$ in Equation A-5 by $\{f_x, f_y\}$. If the displacements and forces due to the virtual-load system are denoted by lower-case letters, then the expressions for these quantities, can be written by replacing the capital letters by the lower-case letters for the forces and displacements, in Equations A-1, A-2 and A-7.

The weight of the three bar truss is given by

$$W = \rho(A_1\ell_1 + A_2\ell_2 + A_3\ell_3) \tag{A-9}$$

The solutions of the problems and the associated algorithms are discussed in subsequent pages. The details of the iterative steps are self explanatory and follow the steps given in Section VIII.

Problem 1. A Single-Displacement Constraint



$E = 10^7$ psi $\qquad \rho = 0.1$ lb/in^3 $\qquad A_{min} = 0.1$ in^2

$\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}_3 = 25000$ psi

$\bar{C}_1 = 0.05''$ $\qquad C_1 = U_x$ or U_y $\qquad C_1 \leq \bar{C}_1$

This problem is solved by using four different algorithms. All algorithms lead to the same minimum-weight design, however, the number of iterations required for each algorithm is not the same. This is a single constrained problem with only one constraint active for all iterations.

Problem 1(a).

The design variables are modified by using the recurrence relation (Equation 43).

$$A_i^{k+i} = A_i^k \left(\lambda_1 \frac{Q_{i1}}{\rho_i \ell_i \bar{A}_i^2} \right)^{1/r}$$

with the step-size parameter $r = 2$. The Lagrange multiplier is determined from the relation (Equation 55),

$$\sqrt{\lambda_1} = \frac{\sum_{i=1}^{n_1} \sqrt{Q_{i1}\rho_i \ell_i}}{\bar{C}_1 - C_1^*}$$

The areas A_1, A_2 and A_3 at the beginning of each iteration are given in rows 1 through 3, and the scaled areas are given in rows 44 through 46. The scaled weight after each iteration is given in row 48. The Lagrange multiplier λ_1 is calculated in row 71.

Row 48 shows that the minimum weight is 113.7 lb and the associated cross-sectional areas are A_1 = 15.07 in^2. A_2 = 0.1 in^2. and A_3 = 0.9318 in^2. The deflection in the x direction at node P is the active constraint, and λ_1 at the optimum is 2141.0. This solution satisfies Equation 56 which gives the relation between λ_1, W_{min}, W^*, $\bar{C}_1$ and C_1^*.

Problem 1(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	Q	.1000E+01	.1644E+02	.1500E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02
2	A_2	.1000E+01	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00
3	A_3	.1000E+01	.1050E+02	.9783E+00	.9383E+00	.9325E+00	.9319E+00	.9318E+00	.9318E+00
4	$\overline{C_1}$	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01
5	E	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
8	$\cos\theta$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta \cdot \cos\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	ℓ_1	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	ℓ_2	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	ℓ_3	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/ℓ_1	.1414E+06	.2324E+07	.2121E+07	.2131E+07	.2132E+07	.2132E+07	.2132E+07	.2132E+07
17	A_2E/ℓ_2	.2000E+06	.2000E+05	.2000E+05	.2000E+05	.2000E+05	.2000E+05	.2000E+05	.2000E+05
18	A_3E/ℓ_3	.1414E+06	.1496E+07	.1384E+06	.1327E+06	.1319E+06	.1318E+06	.1318E+06	.1318E+06
19	$\frac{A_1E}{\ell_1} + \frac{A_3E}{\ell_3}$ (16+17)	.2828E+06	.3820E+07	.2260E+07	.2264E+07	.2264E+07	.2264E+07	.2264E+07	.2264E+07
20	$\frac{A_1E}{\ell_1} - \frac{A_3E}{\ell_3}$ (16-17)	0.	.8284E+06	.1983E+07	.1998E+07	.2000E+07	.2000E+07	.2000E+07	.2000E+07

Problem 1(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.1910E+07	.1130E+07	.1132E+07	.1132E+07	.1132E+07	.1132E+07	.1132E+07
22	$K_2 = (12)(20)$	0.	.4142E+06	.9915E+06	.9992E+06	.9999E+06	.1000E+07	.1000E+07	.1000E+07
23	$K_3 = (10)(19) + 17$	.3414E+06	.1930E+07	.1150E+07	.1152E+07	.1152E+07	.1152E+07	.1152E+07	.1152E+07
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.1516E+12	-.1584E+11	-.1527E+11	-.1519E+11	-.1518E+11	-.1518E+11	-.1518E+11
25	$K_3 \cdot K_2 - K_1 \cdot K_3$ $(23)(22) - (21)(23)$	-.4828E+11	-.3515E+13	-.3161E+12	-.3054E+12	-.3038E+12	-.3036E+12	-.3036E+12	-.3036E+12
26	$U_x = 24/25$	.7071E+00	.4312E-01	.5009E-01	.4999E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.1414E+11	-.1496E+12	-.1384E+11	-.1327E+11	-.1319E+11	-.1318E+11	-.1318E+11	-.1318E+11
28	$U_y = 27/25$	.2929E+00	.4255E-01	.4377E-01	.4345E-01	.4342E-01	.4341E-01	.4341E-01	.4341E-01
29	$U_1 = (26)(9) + (28)(8)$	.7071E+00	.6050E-01	.6637E-01	.6607E-01	.6606E-01	.6605E-01	.6605E-01	.6605E-01
30	$U_2 = U_y\ (28)$	.2929E+00	.4255E-01	.4377E-01	.4345E-01	.4342E-01	.4341E-01	.4341E-01	.4341E-01
31	$U_3 = -(26)(9) + (28)(8)$	-.2929E+00	-.4023E-03	-.4474E-02	-.4630E-02	-.4656E-02	-.4658E-02	-.4659E-02	-.4659E-02
32	$T_1 = \frac{A_1 E}{l_1} U_1\ (16)(29)$	.1000E+06	.1408E+06	.1408E+06	.1408E+06	.1408E+06	.1408E+06	.1408E+06	.1408E+06
33	$T_2 = \frac{A_2 E}{l_2} U_2\ (17)(30)$	.5858E+05	.8511E+03	.8754E+03	.8689E+03	.8683E+03	.8682E+03	.8682E+03	.8682E+03
34	$T_3 = \frac{A_3 E}{l_3} U_3\ (18)(31)$	-.4142E+05	-.6010E+03	-.6190E+03	-.6144E+03	-.6140E+03	-.6139E+03	-.6139E+03	-.6139E+03
35	$\sigma_1 = \frac{T_1}{A_1} = 32/1$	.1000E+06	.8568E+04	.9386E+04	.9344E+04	.9342E+04	.9341E+04	.9341E+04	.9341E+04
36	$\sigma_2 = T_2/A_2 = 33/2$	.5858E+05	.8511E+04	.8754E+04	.8689E+04	.8683E+04	.8682E+04	.8682E+04	.8682E+04
37	$\sigma_3 = T_3/A_3 = 34/3$	-.4142E+05	-.5689E+02	-.6327E+03	-.6549E+03	-.6584E+03	-.6588E+03	-.6589E+03	-.6589E+03
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\bar{\sigma}}$	.4000E+01	.3427E+00	.3755E+00	.3738E+00	.3737E+00	.3736E+00	.3736E+00	.3736E+00
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\bar{\sigma}}$	.2343E+01	.3404E+00	.3501E+00	.3476E+00	.3473E+00	.3473E+00	.3473E+00	.3473E+00
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\bar{\sigma}}$	-.1657E+01	-.2276E-02	-.2531E-01	-.2619E-01	-.2634E-01	-.2635E-01	-.2635E-01	-.2635E-01

Problem 1(a) Cont'd.

No.	Quantity	CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(U_x) = \frac{U_x}{\varepsilon_1} = \frac{26}{4}$	.1414E+02	.8625E+00	.1002E+01	.9999E+00	.1000E+01	.1000E+01	.1000E+01	.1000E+01
42	$\Lambda(U_y) = \frac{U_y}{\varepsilon_1} = \frac{28}{4}$	.5858E+01	.8511E+00	.8754E+00	.8689E+00	.8683E+00	.8682E+00	.8682E+00	.8682E+00
43	$\Lambda(max)$ (38-42)	.1414E+02	.8625E+00	.1002E+01	.9999E+00	.1000E+01	.1000E+01	.1000E+01	.1000E+01
44	$A_1^* = \Lambda \cdot A_1$ (43)(1)	.1414E+02	.1418E+02	.1503E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02
45	$A_2^* = \Lambda \cdot A_2$ (43)(2)	.1414E+02	.8625E-01	.1002E+00	.9999E-01	.1000E+00	.1000E+00	.1000E+00	.1000E+00
46	$A_3^* = \Lambda \cdot A_3$ (43)(3)	.1414E+02	.9123E+01	.9802E+00	.9382E+00	.9325E+00	.9319E+00	.9318E+00	.9318E+00
47	VOLUME $= \sum A_i l_i$	.2707E+04	.1652E+04	.1137E+04	.1137E+04	.1137E+04	.1137E+04	.1137E+04	.1137E+04
48	WT = (47)(ρ)	.2707E+03	.1652E+03	.1137E+03	.1137E+03	.1137E+03	.1137E+03	.1137E+03	.1137E+03
49	f_x	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
50	f_y	0.	0.	0.	0.	0.	0.	0.	0.
51	$K_2 \cdot f_y - K_3 \cdot f_x$ (22)(50)-(23)(49)	-.3414E+06	-.1930E+07	-.1150E+07	-.1152E+07	-.1152E+07	-.1152E+07	-.1152E+07	-.1152E+07
52	$u_x = 51/25$	.7071E-05	.5491E-06	.3638E-05	.3771E-05	.3792E-05	.3794E-05	.3794E-05	.3794E-05
53	$K_2 \cdot f_x - K_1 \cdot f_y$ (22)(49)-(21)(50)	0.	.4142E+06	.9915E+06	.9992E+06	.9999E+06	.1000E+07	.1000E+07	.1000E+07
54	$u_y = 53/25$	0.	-.1178E-06	-.3137E-05	-.3271E-05	-.3292E-05	-.3294E-05	-.3294E-05	-.3294E-05
55	$u_1 = (52)(9) + (54)(8)$	.5000E-05	.3049E-06	.3542E-06	.3535E-06	.3536E-06	.3536E-06	.3536E-06	.3536E-06
56	$u_2 = u_y$ (54)	0.	-.1178E-06	-.3137E-05	-.3271E-05	-.3292E-05	-.3294E-05	-.3294E-05	-.3294E-05
57	$u_3 = -(52)(9) + (54)(8)$	-.5000E-05	-.4716E-06	-.4790E-05	-.4980E-05	-.5009E-05	-.5012E-05	-.5012E-05	-.5012E-05
58	$t_1 = \frac{A_1 u_1 E}{l_1}$ (16)(55)	.7071E+00	.7088E+00	.7515E+00	.7534E+00	.7537E+00	.7537E+00	.7537E+00	.7537E+00
59	$t_2 = \frac{A_2 u_2 E}{l_2}$ (17)(56)	0.	-.2357E-02	-.6273E-01	-.6543E-01	-.6583E-01	-.6588E-01	-.6589E-01	-.6589E-01
60	$t_3 = \frac{A_3 u_3 E}{l_2}$ (18)(57)	-.7071E+00	-.7854E+00	-.6628E+00	-.6608E+00	-.6606E+00	-.6605E+00	-.6605E+00	-.6605E+00

Problem 1(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$Q_{11} = \frac{T_1 t_1 l_1}{E}$	.5000E+00	.7850E+00	.7482E+00	.7501E+00	.7504E+00	.7504E+00	.7504E+00	.7504E+00
62	$Q_{21} = \frac{T_2 t_2 l_2}{E}$	0.	-.1003E-04	-.2746E-03	-.2843E-03	-.2858E-03	-.2860E-03	-.2860E-03	-.2860E-03
63	$Q_{31} = \frac{T_3 t_3 l_3}{E}$	.2871E+00	.3002E-02	.2901E-02	.2871E-02	.2868E-02	.2867E-02	.2867E-02	.2867E-02
64	Passive Element	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01
65	$C_1^* = \sum \frac{Q_{i1}}{A_i}(P)$	0.	-.1163E-03	-.2740E-02	-.2843E-02	-.2858E-02	-.2860E-02	-.2860E-02	-.2860E-02
66	$(Q_{11} \cdot \rho \cdot l_1)^{1/2}$	.1800E+01	.2234E+01	.2300E+01	.2303E+01	.2303E+01	.2304E+01	.2304E+01	.2304E+01
67	$(Q_{21} \cdot \rho \cdot l_2)^{1/2}$	0.	0.	0.	0.	0.	0.	0.	0.
68	$(Q_{31} \cdot \rho \cdot l_3)^{1/2}$	.1210E+01	.1457E+00	.1432E+00	.1425E+00	.1424E+00	.1424E+00	.1424E+00	.1424E+00
69	$66+67+68$	.3090E+01	.2380E+01	.2443E+01	.2446E+01	.2446E+01	.2446E+01	.2446E+01	.2446E+01
70	$\lambda_1^{1/2} = 69/4-65$	.6101E+02	.4740E+02	.4633E+02	.4628E+02	.4627E+02	.4627E+02	.4627E+02	.4627E+02
71	$\lambda_1 = (70)(70)$	.3820E+04	.2255E+04	.2146E+04	.2142E+04	.2141E+04	.2141E+04	.2141E+04	.2141E+04
72	$\lambda_1 \cdot Q_{11}/A_1^{*2} \cdot \rho \cdot l_1$	.1351E+01	.1120E+01	.1005E+01	.1001E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
73	$\lambda_1 \cdot Q_{21}/A_2^{*2} \cdot \rho \cdot l_2$	0.	-.6079E+00	-.1174E+02	-.1218E+02	-.1224E+02	-.1225E+02	-.1225E+02	-.1225E+02
74	$\lambda_1 \cdot Q_{31}/A_3^{*2} \cdot \rho \cdot l_3$	.5595E+00	.1150E-01	.9164E+00	.9880E+00	.9986E+00	.9998E+00	.1000E+01	.1000E+01
75	$A_1^{k+1} = A_1^* \cdot (72)^{1/2}$	.1644E+02	.1500E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02
76	$A_2^{k+1} = A_2^* \cdot (73)^{1/2}$	0.	0.	0.	0.	0.	0.	0.	0.
77	$A_3^{k+1} = A_3^* \cdot (74)^{1/2}$	.1058E+02	.9783E+00	.9383E+00	.9325E+00	.9319E+00	.9318E+00	.9318E+00	.9318E+00
78	A_1	.1644E+02	.1500E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02
79	A_2 (A min)	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00
80	A_3	.1058E+02	.9783E+00	.9383E+00	.9325E+00	.9319E+00	.9318E+00	.9318E+00	.9318E+00

Problem 1(b).

The design variables are modified by using the recurrence relation (Equation 43)

$$A_i^{k+1} = A_i^k \left(\lambda_1 \frac{Q_{i1}}{\rho_i \ell_i A_i^2} \right)_k^{1/r}$$

with the step-size parameter r = 2. The Lagrange multiplier is determined from the relation (Equation 57),

$$\lambda_1 = \frac{\overline{C}_1 - C_1^*}{\sum_{i=1}^{m_1} \frac{Q_{i1}^2}{\rho_i \ell_i A_i^3}}$$

The arrangement of the rows is the same as for Problem 1(a). The number of iterations required to obtain the minimum weight is nearly equal to that for Problem 1(a).

Problem 1(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	A_1	.1000E+01	.1554E+02	.1423E+02	.1509E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02
2	A_2	.1000E+01	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00
3	A_3	.1000E+01	.1000E+02	.9539E+00	.9627E+00	.9340E+00	.9322E+00	.9310E+00	.9310E+00
4	$\overline{C_1}$	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01
5	E	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
8	$\cos\theta$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta\cdot\cos\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	l_1	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	l_2	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	l_3	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/l_1	.1414E+06	.2197E+07	.2012E+07	.2134E+07	.2131E+07	.2132E+07	.2132E+07	.2132E+07
17	A_2E/l_2	.2000E+06	.2000E+05	.2000E+05	.2000E+05	.2000E+05	.2000E+05	.2000E+05	.2000E+05
18	A_3E/l_3	.1414E+06	.1414E+07	.1349E+06	.1361E+06	.1321E+06	.1318E+06	.1318E+06	.1318E+06
19	$\frac{A_1E}{l_1}+\frac{A_3E}{l_3}$ (16+17)	.2828E+06	.3612E+07	.2147E+07	.2270E+07	.2263E+07	.2264E+07	.2264E+07	.2264E+07
20	$\frac{A_1E}{l_1}-\frac{A_3E}{l_3}$ (16-17)	0.	.7832E+06	.1877E+07	.1998E+07	.1999E+07	.2000E+07	.2000E+07	.2000E+07

Problem 1(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.1006E+07	.1073E+07	.1135E+07	.1132E+07	.1132E+07	.1132E+07	.1132E+07
22	$K_2 = (12)(20)$	0.	.3916E+06	.9385E+06	.9989E+06	.9994E+06	.1000E+07	.1000E+07	.1000E+07
23	$K_3 = (10)(19) + 17$	.3414E+06	.1026E+07	.1093E+07	.1155E+07	.1152E+07	.1152E+07	.1152E+07	.1152E+07
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.1434E+12	-.1549E+11	-.1561E+11	-.1521E+11	-.1518E+11	-.1518E+11	-.1518E+11
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4828E+11	-.3144E+13	-.2929E+12	-.3132E+12	-.3041E+12	-.3037E+12	-.3036E+12	-.3035E+12
26	$U_x = 24/25$	.7071E+00	.4562E-01	.5289E-01	.4985E-01	.5001E-01	.5000E-01	.5000E-01	.5000E-01
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.1414E+11	-.1414E+12	-.1349E+11	-.1361E+11	-.1321E+11	-.1318E+11	-.1318E+11	-.1318E+11
28	$U_y = 27/25$	.2929E+00	.4499E-01	.4606E-01	.4346E-01	.4344E-01	.4341E-01	.4341E-01	.4341E-01
29	$U_1 = (26)(9) + (28)(9)$	.7071E+00	.6487E-01	.6997E-01	.6598E-01	.6608E-01	.6605E-01	.6605E-01	.6605E-01
30	$U_2 = U_y$ (28)	.2929E+00	.4499E-01	.4606E-01	.4346E-01	.4344E-01	.4341E-01	.4341E-01	.4341E-01
31	$U_3 = -(26)(9) + (28)(9)$	-.2929E+00	-.4499E-03	-.4828E-02	-.4515E-02	-.4651E-02	-.4657E-02	-.4659E-02	-.4659E-02
32	$T_1 = \frac{A_1 E}{l_1} U_1$ (16)(29)	.1000E+06	.1400E+06	.1400E+06	.1400E+06	.1400E+06	.1400E+06	.1400E+06	.1400E+06
33	$T_2 = \frac{A_2 E}{l_2} U_2$ (17)(30)	.5858E+05	.8997E+03	.9212E+03	.8693E+03	.8687E+03	.8683E+03	.8682E+03	.8682E+03
34	$T_3 = \frac{A_3 E}{l_3} U_3$ (18)(31)	-.4142E+05	-.6362E+03	-.6514E+03	-.6147E+03	-.6143E+03	-.6140E+03	-.6139E+03	-.6139E+03
35	$\sigma_1 = \frac{T_1}{A_1} = 32/1$	.1000E+06	.9061E+04	.9895E+04	.9332E+04	.9345E+04	.9341E+04	.9341E+04	.9341E+04
36	$\sigma_2 = T_2/A_2 = 33/2$	.5858E+05	.8997E+04	.9212E+04	.8693E+04	.8687E+04	.8683E+04	.8682E+04	.8682E+04
37	$\sigma_3 = T_3/A_3 = 34/3$	-.4142E+05	-.6362E+02	-.6829E+03	-.6385E+03	-.6577E+03	-.6586E+03	-.6589E+03	-.6589E+03
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\bar{\sigma}}$	.4000E+01	.3624E+00	.3958E+00	.3733E+00	.3738E+00	.3737E+00	.3737E+00	.3737E+00
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\bar{\sigma}}$	.2343E+01	.3599E+00	.3685E+00	.3477E+00	.3475E+00	.3473E+00	.3473E+00	.3473E+00
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\bar{\sigma}}$	-.1657E+01	-.2545E-02	-.2731E-01	-.2554E-01	-.2631E-01	-.2635E-01	-.2635E-01	-.2636E-01

Problem 1(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(U_x)=\frac{U_x}{\bar{\epsilon}_1}=\frac{26}{4}$	.1414E+02	.9125E+00	.1050E+01	.9970E+00	.1000E+01	.1000E+01	.1000E+01	.1000E+01
42	$\Lambda(U_y)=\frac{U_y}{\bar{\epsilon}_1}=\frac{28}{4}$	.5050E+01	.8997E+00	.9212E+00	.8693E+00	.8687E+00	.8683E+00	.8682E+00	.8682E+00
43	$\Lambda(\max)$ (38-42)	.1414E+02	.9125E+00	.1050E+01	.9970E+00	.1000E+01	.1000E+01	.1000E+01	.1000E+01
44	$A_1^* = \Lambda \cdot A_1$ (43)(1)	.1414E+02	.1410E+02	.1505E+02	.1504E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02
45	$A_2^* = \Lambda \cdot A_2$ (43)(2)	.1414E+02	.9125E-01	.1050E+00	.9970E-01	.1000E+00	.1000E+00	.1000E+00	.1000E+00
46	$A_3^* = \Lambda \cdot A_3$ (43)(3)	.1414E+02	.9124E+01	.1009E+01	.9590E+00	.9342E+00	.9322E+00	.9318E+00	.9318E+00
47	VOLUME = $\sum A_i l_i$	.2707E+04	.1652E+04	.1141E+04	.1137E+04	.1137E+04	.1137E+04	.1137E+04	.1137E+04
48	WT = (47)(ρ)	.2707E+03	.1652E+03	.1141E+03	.1137E+03	.1137E+03	.1137E+03	.1137E+03	.1137E+03
49	f_x	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
50	f_y	0.	0.	0.	0.	0.	0.	0.	0.
51	$K_2 \cdot f_y - K_3 \cdot f_x$ (22)(50)-(23)(49)	-.3414E+06	-.1026E+07	-.1093E+07	-.1155E+07	-.1152E+07	-.1152E+07	-.1152E+07	-.1152E+07
52	$u_x = 51/25$	.7071E-05	.5000E-06	.3733E-05	.3680E-05	.3787E-05	.3793E-05	.3794E-05	.3794E-05
53	$K_2 \cdot f_x - K_1 \cdot f_y$ (22)(49)-(21)(50)	0.	.3916E+06	.9305E+06	.9989E+06	.9994E+06	.1000E+07	.1000E+07	.1000E+07
54	$u_y = 53/25$	0.	-.1246E-06	-.3204E-05	-.3189E-05	-.3287E-05	-.3293E-05	-.3294E-05	-.3294E-05
55	u_1 = (52)(9)+(54)(8)	.5000E-05	.3226E-06	.3740E-06	.3525E-06	.3536E-06	.3536E-06	.3534E-06	.3536E-06
56	$u_2 = u_y$ (54)	0.	-.1246E-06	-.3204E-05	-.3189E-05	-.3287E-05	-.3293E-05	-.3294E-05	-.3294E-05
57	u_3 = -(52)(9)+(54)(8)	-.5000E-05	-.4988E-06	-.4905E-05	-.4863E-05	-.5002E-05	-.5011E-05	-.5012E-05	-.5012E-05
58	$t_1 = \frac{A_1 u_1 E}{l_1}$ (16)(55)	.7071E+00	.7089E+00	.7524E+00	.7522E+00	.7536E+00	.7537E+00	.7537E+00	.7537E+00
59	$t_2 = \frac{A_2 u_2 E}{l_2}$ (17)(56)	0.	-.2491E-02	-.6408E-01	-.6378E-01	-.6573E-01	-.6585E-01	-.6588E-01	-.6589E-01
60	$t_3 = \frac{A_3 u_3 E}{l_3}$ (18)(57)	-.7071E+00	-.7853E+00	-.6618E+00	-.6620E+00	-.6606E+00	-.6605E+00	-.6605E+00	-.6605E+00

Problem 1(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$Q_{11} = \frac{T_1 t_1 l_1}{E}$	.5000E+00	.7857E+00	.7490E+00	.7489E+00	.7503E+00	.7504E+00	.7504E+00	.7504E+00
62	$Q_{21} = \frac{T_2 t_2 l_2}{E}$	0.	-.1121E-04	-.2952E-03	-.2772E-03	-.2855E-03	-.2859E-03	-.2860E-03	-.2860E-03
63	$Q_{31} = \frac{T_3 t_3 l_3}{E}$	.2071E+00	.3173E-02	.3040E-02	.2877E-02	.2869E-02	.2868E-02	.2867E-02	.2867E-02
64	Passive Element	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01
65	$C_1^* = \sum \frac{Q_{i1}}{\lambda_i}(P)$	0.	-.1220E-03	-.2791E-02	-.2781E-02	-.2854E-02	-.2859E-02	-.2860E-02	-.2860E-02
66	$Q_{11}^2 / \rho \cdot l_1 \cdot A_1^{*3}$	.1250E-04	.2471E-04	.2328E-04	.2338E-04	.2326E-04	.2325E-04	.2325E-04	.2325E-04
67	$Q_{21}^2 / \rho \cdot l_2 \cdot A_2^{*3}$	0.	0.	0.	0.	0.	0.	0.	0.
68	$Q_{31}^2 / \rho \cdot l_3 \cdot A_3^{*3}$	.2145E-05	.1074E-08	.1279E-05	.1324E-05	.1428E-05	.1436E-05	.1437E-05	.1437E-05
69	66+67+68	.1465E-04	.2472E-04	.2456E-04	.2462E-04	.2468E-04	.2469E-04	.2469E-04	.2469E-04
70	$\bar{C}_1 - C_1^*$	.5000E-01	.5812E-01	.5279E-01	.5278E-01	.5285E-01	.5286E-01	.5286E-01	.5286E-01
71	$\lambda_1 = 70/69$	.3414E+04	.2028E+04	.2150E+04	.2144E+04	.2141E+04	.2141E+04	.2141E+04	.2141E+04
72	$\lambda_1 \cdot Q_{11} / A_1^{*2} \cdot \rho \cdot l_1$	.1207E+01	.1007E+01	.1005E+01	.1003E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
73	$\lambda_1 \cdot Q_{21} / A_2^{*2} \cdot \rho \cdot l_2$	0.	-.5460E+00	-.1134E+02	-.1196E+02	-.1222E+02	-.1224E+02	-.1225E+02	-.1225E+02
74	$\lambda_1 \cdot Q_{31} / A_3^{*2} \cdot \rho \cdot l_3$	.5000E+00	.1093E-01	.9101E+00	.9470E+00	.9957E+00	.9993E+00	.9999E+00	.1000E+01
75	$A_1^{k+1} = A_1^* \cdot (72)^{1/2}$	.1554E+02	.1423E+02	.1509E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02
76	$A_2^{k+1} = A_2^* \cdot (73)^{1/2}$	0.	0.	0.	0.	0.	0.	0.	0.
77	$A_3^{k+1} = A_3^* \cdot (74)^{1/2}$	.1000E+02	.9539E+00	.9627E+00	.9340E+00	.9322E+00	.9318E+00	.9318E+00	.9318E+00
78	A_1	.1554E+02	.1423E+02	.1509E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02	.1507E+02
79	A_2 (A min)	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00
80	A_3	.1000E+02	.9339E+00	.9627E+00	.9340E+00	.9322E+00	.9318E+00	.9318E+00	.9318E+00

Problem 1(c).

The design variables are modified by using the recurrence relation (Equation 45),

$$A_i^{k+1} = A_i^k \left(1 + \frac{1}{r}\left(\lambda_1 \frac{Q_{i1}}{A_i^2 \rho_i \ell_i} - 1\right)\right)$$

with the step size parameter $r = 2$. The Lagrange multiplier is determined from the relation (Equation 57)

$$\lambda_1 = \frac{\overline{C}_1 - C_1^*}{\sum\limits_{i=1}^{m_1} \frac{Q_{i1}^2}{\rho_i \ell_i A_i^3}}$$

The convergence is slower than for Problems 1(a) or 1(b). After eight iterations the minimum weight is 113.7 lb, but comparing the areas of the members with Problem 1(a) shows that the solution has not as yet converged. An additional 2 or 3 iterations are needed.

Problem 1(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	A_1	.1000E+01	.1561E+02	.1603E+02	.1554E+02	.1441E+02	.1456E+02	.1485E+02	.1503E+02
2	A_2	.1000E+01	.7071E+01	.2649E+01	.5000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00
3	A_3	.1000E+01	.1061E+02	.7473E+01	.4973E+01	.2858E+01	.1600E+01	.1101E+01	.9620E+00
4	$\overline{C_1}$	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01
5	E	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
8	$\cos\theta$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta \cdot \cos\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	ℓ_1	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	ℓ_2	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	ℓ_3	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/ℓ_1	.1414E+06	.2207E+07	.2267E+07	.2190E+07	.2038E+07	.2059E+07	.2101E+07	.2126E+07
17	A_2E/ℓ_2	.2000E+06	.1414E+07	.5698E+06	.1160E+06	.2000E+05	.2000E+05	.2000E+05	.2000E+05
18	A_3E/ℓ_3	.1414E+06	.1500E+07	.1057E+07	.7033E+06	.4041E+06	.2263E+06	.1557E+06	.1360E+06
19	$\frac{A_1E}{\ell_1}+\frac{A_3E}{\ell_3}$ (16+17)	.2828E+06	.3707E+07	.3324E+07	.2902E+07	.2442E+07	.2286E+07	.2257E+07	.2262E+07
20	$\frac{A_1E}{\ell_1}-\frac{A_3E}{\ell_3}$ (16-17)	0.	.7071E+06	.1210E+07	.1495E+07	.1634E+07	.1873E+07	.1945E+07	.1990E+07

Problem 1(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.1854E+07	.1662E+07	.1451E+07	.1221E+07	.1143E+07	.1128E+07	.1131E+07
22	$K_2 = (12)(20)$	0.	.3536E+06	.6850E+06	.7474E+06	.8170E+06	.9165E+06	.9725E+06	.9950E+06
23	$K_3 = (10)(19) + 17$	.3414E+06	.3268E+07	.2232E+07	.1567E+07	.1241E+07	.1163E+07	.1148E+07	.1151E+07
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.2914E+12	-.1627E+12	-.8193E+11	-.4241E+11	-.2463E+11	-.1757E+11	-.1560E+11
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4828E+11	-.5932E+13	-.3343E+13	-.1714E+13	-.8481E+12	-.4088E+12	-.3498E+12	-.3119E+12
26	$U_x = {}^{24}/_{25}$	.7071E+00	.4913E-01	.4866E-01	.4779E-01	.5001E-01	.5038E-01	.5025E-01	.5004E-01
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.1414E+11	-.1500E+12	-.1057E+12	-.7033E+11	-.4041E+11	-.2263E+11	-.1557E+11	-.1360E+11
28	$U_y = {}^{27}/_{25}$	.2929E+00	.2529E-01	.3162E-01	.4103E-01	.4765E-01	.4629E-01	.4453E-01	.4362E-01
29	$U_1 = (26)(9) + (28)(8)$	.7071E+00	.5262E-01	.5676E-01	.6200E-01	.6986E-01	.6838E-01	.6702E-01	.6623E-01
30	$U_2 = U_y\ (28)$	.2929E+00	.2529E-01	.3162E-01	.4103E-01	.4765E-01	.4629E-01	.4453E-01	.4362E-01
31	$U_3 = -(26)(9) + (28)(8)$	-.2929E+00	-.1686E-01	-.1205E-01	-.4785E-02	-.1668E-02	-.2893E-02	-.4043E-02	-.4535E-02
32	$T_1 = \frac{A_1 E}{l_1} U_1\ (16)(29)$	.1000E+06	.1161E+06	.1287E+06	.1381E+06	.1407E+06	.1400E+06	.1408E+06	.1408E+06
33	$T_2 = \frac{A_2 E}{l_2} U_2\ (17)(30)$	.5858E+05	.3576E+05	.1801E+05	.4759E+04	.9530E+03	.9258E+03	.8906E+03	.8724E+03
34	$T_3 = \frac{A_3 E}{l_3} U_3\ (18)(31)$	-.4142E+05	-.2529E+05	-.1274E+05	-.3365E+04	-.6739E+03	-.6546E+03	-.6297E+03	-.6169E+03
35	$\sigma_1 = \frac{T_1}{A_1} = {}^{32}/_{1}$	.1000E+06	.7441E+04	.8028E+04	.8882E+04	.9766E+04	.9667E+04	.9478E+04	.9366E+04
36	$\sigma_2 = {}^{T_2}/_{A_2} = {}^{33}/_{2}$	.5858E+05	.5057E+04	.6323E+04	.8205E+04	.9530E+04	.9258E+04	.8906E+04	.8724E+04
37	$\sigma_3 = {}^{T_3}/_{A_3} = {}^{34}/_{3}$	-.4142E+05	-.2384E+04	-.1705E+04	-.6766E+03	-.2358E+03	-.4091E+03	-.5718E+03	-.6413E+03
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\bar{\sigma}}$	.4000E+01	.2977E+00	.3211E+00	.3553E+00	.3907E+00	.3867E+00	.3791E+00	.3746E+00
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\bar{\sigma}}$	.2343E+01	.2023E+00	.2529E+00	.3282E+00	.3812E+00	.3703E+00	.3562E+00	.3490E+00
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\bar{\sigma}}$	-.1657E+01	-.9536E-01	-.6818E-01	-.2707E-01	-.9433E-02	-.1637E-01	-.2287E-01	-.2565E-01

Problem 1(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(U_x) = \frac{U_x}{c_1} = \frac{26}{4}$	.1414E+02	.9825E+00	.9732E+00	.9558E+00	.1000E+01	.1000E+01	.1005E+01	.1001E+01
42	$\Lambda(U_y) = \frac{U_y}{c_1} = \frac{28}{4}$	.5058E+01	.5857E+00	.6323E+00	.8285E+00	.9530E+00	.9258E+00	.8906E+00	.8724E+00
43	$\Lambda(\max)$ (38-42)	.1414E+02	.9825E+00	.9732E+00	.9558E+00	.1000E+01	.1000E+01	.1005E+01	.1001E+01
44	$A_1^* = \Lambda \cdot A_1$ (43)(1)	.1414E+02	.1533E+02	.1560E+02	.1486E+02	.1441E+02	.1467E+02	.1493E+02	.1504E+02
45	$A_2^* = \Lambda \cdot A_2$ (43)(2)	.1414E+02	.6948E+01	.2773E+01	.5544E+00	.1000E+00	.1000E+00	.1005E+00	.1001E+00
46	$A_3^* = \Lambda \cdot A_3$ (43)(3)	.1414E+02	.1042E+02	.7273E+01	.4754E+01	.2858E+01	.1612E+01	.1107E+01	.9626E+00
47	VOLUME = $\Sigma A_i l_i$	.2707E+04	.2169E+04	.1756E+04	.1414E+04	.1226E+04	.1157E+04	.1139E+04	.1137E+04
48	WT = (47)(ρ)	.2707E+03	.2169E+03	.1756E+03	.1414E+03	.1226E+03	.1157E+03	.1139E+03	.1137E+03
49	f_x	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
50	f_y	0.	0.	0.	0.	0.	0.	0.	0.
51	$K_2 \cdot f_y - K_3 \cdot f_x$ (22)(50)−(23)(49)	-.3414E+06	-.3268E+07	-.2232E+07	-.1567E+07	-.1241E+07	-.1163E+07	-.1148E+07	-.1151E+07
52	$u_x = 51/25$	.7071E-05	.5509E-06	.6676E-06	.9139E-06	.1463E-05	.2379E-05	.3283E-05	.3691E-05
53	$K_3 \cdot f_x - K_1 \cdot f_y$ (23)(49)−(21)(50)	0.	.3536E+06	.6050E+06	.7474E+06	.8170E+06	.9165E+06	.9725E+06	.9950E+06
54	$u_y = 53/25$	0.	-.5960E-07	-.1810E-06	-.4360E-06	-.9633E-06	-.1875E-05	-.2781E-05	-.3191E-05
55	u_1 = (52)(9)+(54)(8)	.5000E-05	.3474E-06	.3441E-06	.3379E-06	.3536E-06	.3562E-06	.3553E-06	.3538E-06
56	$u_2 = u_y$ (54)	0.	-.5960E-07	-.1810E-06	-.4360E-06	-.9633E-06	-.1875E-05	-.2781E-05	-.3191E-05
57	u_3 = −(52)(9)+(54)(8)	-.5000E-05	-.4317E-06	-.6000E-06	-.9545E-06	-.1716E-05	-.3008E-05	-.4288E-05	-.4866E-05
58	$t_1 = \frac{A_1 u_1 E}{l_1}$ (16)(55)	.7071E+00	.7667E+00	.7600E+00	.7429E+00	.7207E+00	.7336E+00	.7464E+00	.7522E+00
59	$t_2 = \frac{A_2 u_2 E}{l_2}$ (17)(56)	0.	-.8429E-01	-.1031E+00	-.5857E-01	-.1927E-01	-.3750E-01	-.5561E-01	-.6381E-01
60	$t_3 = \frac{A_3 u_3 E}{l_3}$ (18)(57)	-.7071E+00	-.6475E+00	-.6342E+00	-.6713E+00	-.6935E+00	-.6806E+00	-.6678E+00	-.6620E+00

Problem 1(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$Q_{11} = \frac{T_1 t_1 l_1}{E}$	.5000E+00	.6296E+00	.7098E+00	.7252E+00	.7173E+00	.7302E+00	.7431E+00	.7490E+00
62	$Q_{21} = \frac{T_2 t_2 l_2}{E}$	0.	-.1507E-01	-.9209E-02	-.1203E-02	-.9181E-04	-.1736E-03	-.2476E-03	-.2784E-03
63	$Q_{31} = \frac{T_3 t_3 l_3}{E}$	.2071E+00	.1150E+00	.5712E-01	.1597E-01	.3305E-02	.3150E-02	.2974E-02	.2888E-02
64	Passive Element	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01
65	$C_1^* = \sum \frac{Q_{i1}}{\lambda_i}(P)$	0.	0.	0.	0.	-.9179E-03	-.1723E-02	-.2464E-02	-.2782E-02
66	$Q_{11}^2 / \rho \cdot l_1 \cdot A_1^{*3}$	.1250E-04	.1559E-04	.1876E-04	.2268E-04	.2429E-04	.2387E-04	.2347E-04	.2338E-04
67	$Q_{21}^2 / \rho \cdot l_2 \cdot A_2^{*3}$	0.	.1355E-06	.8096E-06	.1700E-05	0.	0.	0.	0.
68	$Q_{31}^2 / \rho \cdot l_3 \cdot A_3^{*3}$	.2145E-05	.1675E-05	.1199E-05	.3360E-06	.6614E-07	.3350E-06	.9225E-06	.1322E-05
69	66+67+68	.1465E-04	.1736E-04	.2077E-04	.2471E-04	.2436E-04	.2421E-04	.2440E-04	.2462E-04
70	$\overline{C}_1 - C_1^*$	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5092E-01	.5172E-01	.5246E-01	.5278E-01
71	$\lambda_1 = 70/69$	.3414E+04	.2880E+04	.2407E+04	.2023E+04	.2090E+04	.2136E+04	.2151E+04	.2144E+04
72	$\lambda_1 \cdot Q_{11} / A_1^{*2} \cdot \rho \cdot l_1$	.1207E+01	.1091E+01	.9927E+00	.9400E+00	.1020E+01	.1025E+01	.1014E+01	.1003E+01
73	$\lambda_1 \cdot Q_{21} / A_2^{*2} \cdot \rho \cdot l_2$	0.	-.1799E+00	-.5816E+00	-.1584E+01	-.3836E+01	-.7305E+01	-.1055E+02	-.1192E+02
74	$\lambda_1 \cdot Q_{31} / A_3^{*2} \cdot \rho \cdot l_3$	.5000E+00	.4342E+00	.3676E+00	.2023E+00	.1196E+00	.3662E+00	.7383E+00	.9449E+00
75	$A_1^{k+1} = A_1^k(1+0.5(\underline{72}-1))$	.1561E+02	.1603E+02	.1554E+02	.1441E+02	.1456E+02	.1485E+02	.1503E+02	.1507E+02
76	$A_2^{k+1} = A_2^*(1+0.5(\underline{73}-1))$	.7071E+01	.2849E+01	.5000E+00	-.1628E+00	-.1418E+00	-.3177E+00	-.4797E+00	-.5464E+00
77	$A_3^{k+1} = A_3^*(1+0.5(\underline{74}-1))$	.1061E+02	.7473E+01	.4973E+01	.2858E+01	.1600E+01	.1101E+01	.9620E+00	.9361E+00
78	A_1	.1561E+02	.1603E+02	.1554E+02	.1441E+02	.1456E+02	.1485E+02	.1503E+02	.1507E+02
79	A_2	.7071E+01	.2849E+01	.5000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00
80	A_3	.1061E+02	.7473E+01	.4973E+01	.2858E+01	.1600E+01	.1101E+01	.9620E+00	.9361E+00

Problem 1(d).

The design variables are modified by using the recurrence relation (Equation 158),

$$A_i^{k+1} = \frac{A_i^k}{\left(1 - \frac{1}{r}\left(\lambda_1 \frac{Q_{i1}}{\rho_i \ell_i A_i^2} - 1\right)\right)}$$

with the step-size parameter $r = 2$. The Lagrange multiplier is determined from the relation (Equation 57)

$$\lambda_1 = \frac{\bar{C}_1 - C_1^*}{\sum_{i=1}^{m_1} \frac{Q_{i1}^2}{\rho_i \ell_i A_i^3}}$$

This algorithm is equivalent to the algorithm based on the use of the reciprocal design variable. The convergence for this algorithm is slower than for all the three previous problems. After eight iterations the minimum weight is 123.0 lb and needs additional iterations for the solution to coverge to the minimum.

Problem 1(d) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	A_1	.1000E+01	.1578E+02	.1618E+02	.1618E+02	.1585E+02	.1529E+02	.1467E+02	.1413E+02
2	A_2	.1000E+01	.9428E+01	.5855E+01	.3485E+01	.1944E+01	.9912E+00	.4486E+00	.1739E+00
3	A_3	.1000E+01	.1131E+02	.8733E+01	.6813E+01	.5343E+01	.4183E+01	.3245E+01	.2477E+01
4	$\overline{C_1}=\overline{C_2}$	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01
5	E	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
8	$\cos\theta$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta\cdot\cos\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	ℓ_1	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	ℓ_2	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	ℓ_3	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/ℓ_1	.1414E+06	.2231E+07	.2289E+07	.2288E+07	.2241E+07	.2163E+07	.2074E+07	.1998E+07
17	A_2E/ℓ_2	.2000E+06	.1886E+07	.1171E+07	.6971E+06	.3889E+06	.1982E+06	.8971E+05	.3478E+05
18	A_3E/ℓ_3	.1414E+06	.1600E+07	.1235E+07	.9634E+06	.7556E+06	.5915E+06	.4589E+06	.3504E+06
19	$\frac{A_1E}{\ell_1}+\frac{A_3E}{\ell_3}$ (16+17)	.2828E+06	.3831E+07	.3524E+07	.3252E+07	.2997E+07	.2754E+07	.2573E+07	.2349E+07
20	$\frac{A_1E}{\ell_1}-\frac{A_3E}{\ell_3}$ (16−17)	0.	.6310E+06	.1854E+07	.1325E+07	.1486E+07	.1571E+07	.1615E+07	.1648E+07

Problem 1(d) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.1916E+07	.1762E+07	.1626E+07	.1498E+07	.1377E+07	.1267E+07	.1174E+07
22	$K_2 = (12)(20)$	0.	.3155E+06	.5268E+06	.6623E+06	.7428E+06	.7857E+06	.8077E+06	.8239E+06
23	$K_3 = (10)(19) + 17$	.3414E+06	.3881E+07	.2933E+07	.2323E+07	.1887E+07	.1575E+07	.1356E+07	.1209E+07
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.3486E+12	-.2486E+12	-.1661E+12	-.1144E+12	-.7898E+11	-.5486E+11	-.3852E+11
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4828E+11	-.7182E+13	-.4898E+13	-.3338E+13	-.2276E+13	-.1552E+13	-.1065E+13	-.7409E+12
26	$U_x = 24/25$	.7071E+00	.4854E-01	.4928E-01	.4975E-01	.5028E-01	.5087E-01	.5149E-01	.5198E-01
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.1414E+11	-.1600E+12	-.1235E+12	-.9634E+11	-.7556E+11	-.5915E+11	-.4589E+11	-.3504E+11
28	$U_y = 27/25$	.2929E+00	.2228E-01	.2526E-01	.2887E-01	.3320E-01	.3818E-01	.4307E-01	.4729E-01
29	$U_1 = (26)(9) + (28)(8)$	.7071E+00	.5007E-01	.5265E-01	.5559E-01	.5983E-01	.6291E-01	.6686E-01	.7019E-01
30	$U_2 = U_y \ (28)$	.2929E+00	.2228E-01	.2526E-01	.2887E-01	.3328E-01	.3810E-01	.4307E-01	.4729E-01
31	$U_3 = -(26)(9) + (28)(8)$	-.2929E+00	-.1057E-01	-.1693E-01	-.1477E-01	-.1208E-01	-.9829E-02	-.5954E-02	-.3319E-02
32	$T_1 = \frac{A_1 E}{l_1} U_1 \ (16)(29)$	.1000E+06	.1117E+06	.1205E+06	.1272E+06	.1323E+06	.1361E+06	.1387E+06	.1403E+06
33	$T_2 = \frac{A_2 E}{l_2} U_2 \ (17)(30)$	.5858E+05	.4201E+05	.2958E+05	.2012E+05	.1291E+05	.7553E+04	.3864E+04	.1645E+04
34	$T_3 = \frac{A_3 E}{l_3} U_3 \ (18)(31)$	-.4142E+05	-.2971E+05	-.2091E+05	-.1423E+05	-.9129E+04	-.5341E+04	-.2732E+04	-.1163E+04
35	$\sigma_1 = \frac{T_1}{A_1} = 32/1$	.1000E+06	.7082E+04	.7446E+04	.7862E+04	.8348E+04	.8897E+04	.9456E+04	.9927E+04
36	$\sigma_2 = T_2/A_2 = 33/2$	.5858E+05	.4456E+04	.5051E+04	.5773E+04	.6640E+04	.7620E+04	.8614E+04	.9457E+04
37	$\sigma_3 = T_3/A_3 = 34/3$	-.4142E+05	-.2626E+04	-.2395E+04	-.2089E+04	-.1709E+04	-.1277E+04	-.8420E+03	-.4694E+03
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\bar{\sigma}}$	.4000E+01	.2833E+00	.2979E+00	.3145E+00	.3339E+00	.3559E+00	.3782E+00	.3971E+00
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\bar{\sigma}}$	.2343E+01	.1782E+00	.2021E+00	.2309E+00	.2656E+00	.3048E+00	.3445E+00	.3783E+00
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\bar{\sigma}}$	-.1657E+01	-.1050E+00	-.9579E-01	-.8354E-01	-.6835E-01	-.5108E-01	-.3368E-01	-.1878E-01

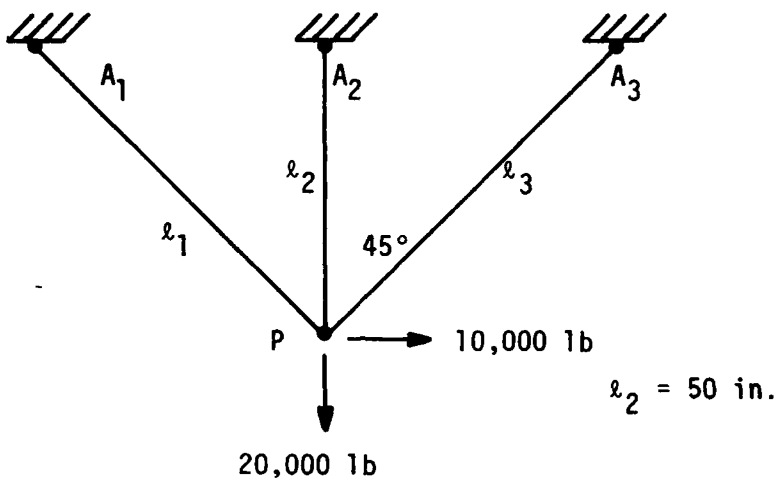
Problem 1(d) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(\sigma_x) = \frac{\sigma_x}{\varepsilon_1} = \frac{26}{4}$	.1414E+02	.9707E+00	.9841E+00	.9950E+00	.1006E+01	.1017E+01	.1030E+01	.1040E+01
42	$\Lambda(\sigma_y) = \frac{\sigma_y}{\varepsilon_1} = \frac{28}{4}$	.5050E+01	.4456E+00	.5051E+00	.5773E+00	.6640E+00	.7620E+00	.8614E+00	.9457E+00
43	$\Lambda(max)$ (38-42)	.1414E+02	.9707E+00	.9841E+00	.9950E+00	.1006E+01	.1017E+01	.1030E+01	.1040E+01
44	$A_1^* = \Lambda \cdot A_1$ (43)(1)	.1414E+02	.1531E+02	.1593E+02	.1610E+02	.1594E+02	.1556E+02	.1510E+02	.1469E+02
45	$A_2^* = \Lambda \cdot A_2$ (43)(2)	.1414E+02	.9152E+01	.5762E+01	.3460E+01	.1955E+01	.1000E+01	.4619E+00	.1000E+00
46	$A_3^* = \Lambda \cdot A_3$ (43)(3)	.1414E+02	.1090E+02	.8594E+01	.6779E+01	.5373E+01	.4256E+01	.3341E+01	.2576E+01
47	VOLUME = $\Sigma A_i \ell_i$	.2707E+04	.2317E+04	.2022E+04	.1791E+04	.1605E+04	.1452E+04	.1327E+04	.1230E+04
48	WT = (47)(ρ)	.2707E+03	.2317E+03	.2022E+03	.1791E+03	.1605E+03	.1452E+03	.1327E+03	.1230E+03
49	f_x	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
50	f_y	0.	0.	0.	0.	0.	0.	0.	0.
51	$K_2 \cdot f_y - K_3 \cdot f_x$ (22)(50)-(23)(49)	-.3414E+06	-.3001E+07	-.2933E+07	-.2323E+07	-.1887E+07	-.1575E+07	-.1356E+07	-.1209E+07
52	$u_x = 51/25$	.7071E-05	.5293E-06	.5998E-06	.6959E-06	.8292E-06	.1015E-05	.1273E-05	.1632E-05
53	$K_2 \cdot f_x - K_1 \cdot f_y$ (22)(49)-(21)(50)	0.	.3155E+06	.5268E+06	.6623E+06	.7428E+06	.7857E+06	.8077E+06	.8239E+06
54	$u_y = 53/25$	0.	-.4393E-07	-.1077E-06	-.1984E-06	-.3264E-06	-.5061E-06	-.7581E-06	-.1112E-05
55	u_1 = (52)(9)+(54)(8)	.5000E-05	.3432E-06	.3479E-06	[illegible]	.3556E-06	.3597E-06	.3641E-06	.3676E-06
56	$u_2 = u_y$ (54)	0.	-.4393E-07	-.1077E-06	-.1984E-06	-.3264E-06	-.5061E-06	-.7581E-06	-.1112E-05
57	u_3 = -(52)(9)+(54)(8)	-.5000E-05	-.4053E-06	-.5803E-06	[illegible]	-.8171E-06	-.1075E-05	-.1436E-05	-.1940E-05
58	$t_1 = \frac{A_1 u_1 E}{\ell_1}$ (16)(55)	.7071E+00	.7657E+00	.7963E+00	.8049E+00	.7969E+00	.7781E+00	.7552E+00	.7345E+00
59	$t_2 = \frac{A_2 u_2 E}{\ell_2}$ (17)(56)	0.	-.8284E-01	-.1262E+00	-.1303E+[illegible]	-.1269E+00	-.1003E+00	-.6801E-01	-.3067E-01
60	$t_3 = \frac{A_3 u_3 E}{\ell_3}$ (18)(57)	-.7071E+00	-.6485E+00	-.6179E+00	-.6093E+00	-.6174E+00	-.6362E+00	-.6590E+00	-.6798E+00

Problem 1(d) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$Q_{11} = \frac{T_1 t_1 l_1}{E}$	.5000E+00	.6049E+00	.6786E+00	.7239E+00	.7454E+00	.7487E+00	.7406E+00	.7284E+00
62	$Q_{21} = \frac{T_2 t_2 l_2}{E}$	0.	-.1740E-01	-.1866E-01	-.1392E-01	-.8192E-02	-.3789E-02	-.1314E-02	-.3180E-03
63	$Q_{31} = \frac{T_3 t_3 l_3}{E}$	.2071E+00	.1362E+00	.9130E-01	.6130E-01	.3985E-01	.2403E-01	.1273E-01	.5590E-02
64	Passive Elements	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01	.2000E+01
65	$C_1^* = \sum \frac{Q_{i1}}{\lambda_i}(P)$	0.	0.	0.	0.	0.	0.	0.	0.
66	$Q_{11}^2/\rho \cdot l_1 \cdot A_1^{*3}$	.1250E-04	.1441E-04	.1612E-04	.1777E-04	.1941E-04	.2104E-04	.2251E-04	.2368E-04
67	$Q_{21}^2/\rho \cdot l_2 \cdot A_2^{*3}$	0.	.7900E-07	.3639E-06	.9286E-06	.1795E-05	.2800E-05	.3503E-05	.3423E-05
68	$Q_{31}^2/\rho \cdot l_3 \cdot A_3^{*3}$	.2145E-05	.1981E-05	.1860E-05	.1706E-05	.1448E-05	.1059E-05	.6146E-06	.2586E-06
69	$66+67+68$	.1465E-04	.1647E-04	.1834E-04	.2040E-04	.2266E-04	.2490E-04	.2663E-04	.2736E-04
70	$\overline{C}_1 - C_1^*$	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01
71	$\lambda_1 = 70/69$	.3414E+04	.3036E+04	.2726E+04	.2451E+04	.2207E+04	.2008E+04	.1878E+04	.1828E+04
72	$\lambda_1 \cdot Q_{11}/A_1^{*2} \cdot \rho \cdot l_1$	.1207E+01	.1100E+01	.1031E+01	.9683E+00	.9160E+00	.8781E+00	.8620E+00	.8726E+00
73	$\lambda_1 \cdot Q_{21}/A_2^{*2} \cdot \rho \cdot l_2$	0.	-.1262E+00	-.3063E+00	-.5672E+00	-.9456E+00	-.1497E+01	-.2312E+01	-.3557E+01
74	$\lambda_1 \cdot Q_{31}/A_3^{*2} \cdot \rho \cdot l_3$	.5000E+00	.4850E+00	.4769E+00	.4624E+00	.4308E+00	.3768E+00	.3028E+00	.2178E+00
75	$A_1^{(k+1)} = A_1^*/(1-0.5(\underline{72}-1))$	.1578E+02	.1618E+02	.1618E+02	.1585E+02	.1529E+02	.1467E+02	.1413E+02	.1381E+02
76	$A_2^{(k+1)} = A_2^*/(1-0.5(\underline{73}-1))$	.9428E+01	.5855E+01	.3485E+01	.1944E+01	.9912E+00	.4486E+00	.1739E+00	.5515E-01
77	$A_3^{(k+1)} = A_3^*/(1-0.5(74-1))$	.1131E+02	.8733E+01	.6813E+01	.5343E+01	.4183E+01	.3245E+01	.2477E+01	.1852E+01
78	A_1	.1578E+02	.1618E+02	.1618E+02	.1585E+02	.1529E+02	.1467E+02	.1413E+02	.1381E+02
79	A_2	.9428E+01	.5855E+01	.3485E+01	.1944E+01	.9912E+00	.4486E+00	.1739E+00	.1000E+00
80	A_3	.1131E+02	.8733E+01	.6813E+01	.5343E+01	.4183E+01	.3245E+01	.2477E+01	.1852E+01

Problem 2. Multiple Displacement Constraints.



$E = 10^7$ psi $\quad \rho = 0.1$ lb/in^3 $\quad A_{min} = 0.1$ in^2

$\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}_3 = 25{,}000$ psi.

$\bar{C}_1 = 0.05$ in. $\quad \bar{C}_2 = 0.05$ in.

$C_1 = U_x \quad C_2 = U_y$

$C_1 \leq \bar{C}_1 \quad C_2 \leq \bar{C}_2$

This problem is solved by using three different algorithms. All algorithms lead nearly to the same minimum-weight design, but the convergence behavior is not the same. The two constraints are active throughout the iterative history.

Problem 2(a).

The design variables are modified by using the recurrence relation (Equation 71)

$$A_i^{k+1} = A_i^k \left(1 + \frac{1}{r}\left(\sum_{j=1}^{2} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} - 1\right)\right)_k$$

with the step-size parameter $r = 2$. The Lagrange multipliers are determined from the relation (Equation 86)

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

where

$$A = \Sigma \frac{Q_{i1}\, Q_{i1}}{\rho_i \ell_i A_i^3} \qquad \text{(Active Elements)}$$

$$B = \Sigma \frac{Q_{i1}\, Q_{i2}}{\rho_i \ell_i A_i^3} \qquad \text{(Active Elements)}$$

$$C = \Sigma \frac{Q_{i2}\, Q_{i2}}{\rho_i \ell_i A_i^3} \qquad \text{(Active Elements)}$$

$$R_1 = 2(C_1^* + B_1 - \overline{C}_1) + B_1$$

$$R_2 = 2(C_2^* + B_2 - \overline{C}_2) + B_2$$

$$C_1^* = \Sigma \frac{Q_{i1}}{A_i} \qquad \text{(Passive)}$$

$$C_2^* = \Sigma \frac{Q_{i2}}{A_i} \qquad \text{(Passive)}$$

$$B_1 = \Sigma \frac{Q_{i1}}{A_i} \qquad \text{(Active)}$$

$$B_2 = \Sigma \frac{Q_{i2}}{A_i} \qquad \text{(Active)}$$

The Lagrange multipliers are given by

$$\lambda_1 = \frac{B \cdot R_2 - C \cdot R_1}{B^2 - AB}$$

$$\lambda_2 = \frac{B \cdot R_1 - A \cdot R_2}{B^2 - AB}$$

where

$$Q_{i1} = \frac{T_i t_i^1 \ell_i}{E} \quad \text{and} \quad Q_{i2} = \frac{T_i t_i^2 \ell_i}{E}$$

t_i^1 and t_i^2 are the forces in the bars dur to a unit load applied at node P in the X and Y direction respectively.

The weight of the structure after eight iterations was 151.1 lb. One more iteration was needed to reach the optimum design. The optimum weight of the structure was 150.7 lb with A_1 = 14.14 in^2, A_2 = 10.10 in^2 and A_3 = 0.1 in^2. For the optimum design, λ_1 = 1000 and λ_2 = 2000. This design satisfies the relation

$$W - W^* = \sum_{j=1}^{m} \lambda_j (\overline{C}_j - C_j^*)$$

which is valid at optimum solution.

Problem 2(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	A_1	.1000E+01	.1475E+02	.1359E+02	.1423E+02	.1411E+02	.1414E+02	.1414E+02	.1414E+02
2	A_2	.1000E+01	.8195E+01	.1047E+02	.9739E+01	.1004E+02	.9995E+01	.1000E+02	.1000E+02
3	A_3	.1000E+01	.7675E+01	.4423E+01	.2388E+01	.1218E+01	.6134E+00	.3068E+00	.1534E+00
4	$\overline{C}_1 = \overline{C}_2$	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01
5	E	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06
8	$\cos\theta$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta \cdot \cos\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	ℓ_1	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	ℓ_2	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	ℓ_3	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	$A_1 E/\ell_1$	.1414E+06	.2086E+07	.1922E+07	.2012E+07	.1995E+07	.2000E+07	.2000E+07	.2000E+07
17	$A_2 E/\ell_2$	.2000E+06	.1639E+07	.2095E+07	.1948E+07	.2007E+07	.1999E+07	.2000E+07	.2000E+07
18	$A_3 E/\ell_3$	.1414E+06	.1085E+07	.6255E+06	.3376E+06	.1723E+06	.8675E+05	.4339E+05	.2170E+05
19	$\frac{A_1 E}{\ell_1} + \frac{A_3 E}{\ell_3}$ (16+17)	.2828E+06	.3171E+07	.2547E+07	.2350E+07	.2167E+07	.2087E+07	.2043E+07	.2022E+07
20	$\frac{A_1 E}{\ell_1} - \frac{A_3 E}{\ell_3}$ (16-17)	0.	.1000E+07	.1296E+07	.1674E+07	.1823E+07	.1914E+07	.1957E+07	.1978E+07

Problem 2(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.1506E+07	.1274E+07	.1175E+07	.1084E+07	.1044E+07	.1022E+07	.1011E+07
22	$K_2 = (12)(20)$	0.	.5802E+06	.6481E+06	.8372E+06	.9114E+06	.9568E+06	.9783E+06	.9892E+06
23	$K_3 = (10)(19) + 17$	.3414E+06	.3225E+07	.3368E+07	.3122E+07	.3091E+07	.3043E+07	.3022E+07	.3011E+07
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.2224E+12	-.2072E+12	-.1448E+12	-.1268E+12	-.1129E+12	-.1065E+12	-.1033E+12
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4828E+11	-.4863E+13	-.3870E+13	-.2967E+13	-.2519E+13	-.2268E+13	-.2130E+13	-.2065E+13
26	$U_x = {}^{24}/_{25}$	.7071E+00	.4574E-01	.5355E-01	.4880E-01	.5034E-01	.4996E-01	.5000E-01	.5000E-01
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.2828E+11	-.2671E+12	-.1899E+12	-.1512E+12	-.1256E+12	-.1130E+12	-.1065E+12	-.1033E+12
28	$U_y = {}^{27}/_{25}$	.5858E+00	.5493E-01	.4987E-01	.5097E-01	.4986E-01	.5002E-01	.5000E-01	.5000E-01
29	$U_1 = (26)(9) + (28)(9)$	.9142E+00	.7118E-01	.7256E-01	.7055E-01	.7085E-01	.7070E-01	.7071E-01	.7071E-01
30	$U_2 = U_y \; (28)$	.5858E+00	.5493E-01	.4987E-01	.5097E-01	.4986E-01	.5002E-01	.5000E-01	.5000E-01
31	$U_3 = -(26)(9) + (28)(9)$	-.8579E-01	.6497E-02	-.3164E-02	.1536E-02	-.3414E-03	.4499E-04	-.2171E-05	.6921E-06
32	$T_1 = \frac{A_1 E}{l_1} U_1 \; (16)(29)$	.1293E+06	.1485E+06	.1394E+06	.1419E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06
33	$T_2 = \frac{A_2 E}{l_2} U_2 \; (17)(30)$	.1172E+06	.9883E+05	.1028E+06	.9927E+05	.1001E+06	.9999E+05	.1000E+06	.1000E+06
34	$T_3 = \frac{A_3 E}{l_3} U_3 \; (18)(31)$	-.1213E+05	.7052E+04	-.1979E+04	.5187E+03	-.5882E+02	.3982E+01	-.9422E-01	.1502E-01
35	$\sigma_1 = \frac{T_1}{A_1} = {}^{32}/_{1}$	.1293E+06	.1007E+05	.1026E+05	.9977E+04	.1002E+05	.9998E+04	.1000E+05	.1000E+05
36	$\sigma_2 = T_2/A_2 = {}^{33}/_{2}$	.1172E+06	.1099E+05	.9814E+04	.1019E+05	.9972E+04	.1000E+05	.1000E+05	.1000E+05
37	$\sigma_3 = T_3/A_3 = {}^{34}/_{3}$	-.1213E+05	.9188E+03	-.4474E+03	.2172E+03	-.4828E+02	.6362E+01	-.3071E+00	.9788E-01
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\bar{\sigma}}$	.5172E+01	.4027E+00	.4105E+00	.3991E+00	.4008E+00	.3999E+00	.4000E+00	.4000E+00
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\bar{\sigma}}$	.4686E+01	.4394E+00	.3926E+00	.4078E+00	.3989E+00	.4002E+00	.4000E+00	.4000E+00
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\bar{\sigma}}$	-.4853E+00	.3675E-01	-.1798E-01	.8698E-02	-.1931E-02	.2545E-03	-.1228E-04	.3915E-05

Problem 2(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(\sigma_x)=\frac{\sigma_x}{\bar{\sigma}_1}=\frac{26}{4}$	.1414E+02	.9140E+00	.1071E+01	.9760E+00	.1007E+01	.9992E+00	.1000E+01	.1000E+01
42	$\Lambda(\sigma_y)=\frac{\sigma_y}{\bar{\sigma}_1}=\frac{28}{4}$	.1172E+02	.1099E+01	.9814E+00	.1019E+01	.9972E+00	.1000E+01	.1000E+01	.1000E+01
43	$\Lambda(max)$ (38-42)	.1414E+02	.1099E+01	.1071E+01	.1019E+01	.1007E+01	.1000E+01	.1000E+01	.1000E+01
44	$A_1^* = \Lambda \cdot A_1$ (43)(1)	.1414E+02	.1620E+02	.1455E+02	.1450E+02	.1420E+02	.1415E+02	.1414E+02	.1414E+02
45	$A_2^* = \Lambda \cdot A_2$ (43)(2)	.1414E+02	.9803E+01	.1122E+02	.9927E+01	.1011E+02	.9999E+01	.1000E+02	.1000E+02
46	$A_3^* = \Lambda \cdot A_3$ (43)(3)	.1414E+02	.8431E+01	.4737E+01	.2434E+01	.1227E+01	.6137E+00	.3068E+00	.1534E+00
47	VOLUME = $\sum A_i l_i$	.2707E+04	.2192E+04	.1925E+04	.1694E+04	.1596E+04	.1544E+04	.1522E+04	.1511E+04
48	WT = (47)(ρ)	.2707E+03	.2192E+03	.1925E+03	.1694E+03	.1596E+03	.1544E+03	.1572E+03	.1511E+03
49	f_x'	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
50	f_y'	0.	0.	0.	0.	0.	0.	0.	0.
51	$K_2 \cdot f_y' - K_3 \cdot f_x'$ (22)(50)-(23)(49)	-.3414E+06	-.3225E+07	-.3360E+07	-.3122E+07	-.3091E+07	-.3043E+07	-.3022E+07	-.3011E+07
52	$u_x' = 51/25$	.7071E-05	.6631E-06	.8784E-06	.1052E-05	.1227E-05	.1347E-05	.1419E-05	.1458E-05
53	$K_2 \cdot f_x' - K_1 \cdot f_y'$ (22)(49)-(21)(50)	0.	.5002E+06	.6481E+06	.8372E+06	.9114E+06	.9568E+06	.9783E+06	.9892E+06
54	$u_y' = 53/25$	0.	-.1029E-06	-.1675E-06	-.2821E-06	-.3618E-06	-.4235E-06	-.4592E-06	-.4790E-06
55	$u_1' = (52)(9)+(54)(8)$	.5000E-05	.3962E-06	.4970E-06	.5446E-06	.6118E-06	.6527E-06	.6783E-06	.6922E-06
56	$u_2' = u_y$ (54)	0.	-.1029E-06	-.1675E-06	-.2821E-06	-.3618E-06	-.4235E-06	-.4592E-06	-.4790E-06
57	$u_3' = -(52)(9)+(54)(8)$	-.5000E-05	-.5416E-06	-.7339E-06	-.9435E-06	-.1123E-05	-.1252E-05	-.1328E-05	-.1370E-05
58	$t_1' = \frac{A_1 u_1' E}{l_1}$ (16)(55)	.7071E+00	.8263E+00	.9552E+00	.1096E+01	.1221E+01	.1306E+01	.1357E+01	.1385E+01
59	$t_2' = \frac{A_2 u_2' E}{l_2}$ (17)(56)	0.	-.1686E+00	-.3508E+00	-.5495E+00	-.7262E+00	-.8465E+00	-.9185E+00	-.9580E+00
60	$t_3' = \frac{A_3 u_3' E}{l_3}$ (18)(57)	-.7071E+00	-.5879E+00	-.4590E+00	-.3186E+00	-.1936E+00	-.1086E+00	-.5762E-01	-.2972E-01

Problem 2(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$Q_{11} = \frac{T_1 t_1^1 l_1}{E}$	.6465E+00	.8675E+00	.9418E+00	.1100E+01	.1220E+01	.1306E+01	.1357E+01	.1385E+01
62	$Q_{21} = \frac{T_2 t_2^1 l_2}{E}$	0.	-.7589E-01	-.1803E+00	-.2727E+00	-.3634E+00	-.4232E+00	-.4593E+00	-.4790E+00
63	$Q_{31} = \frac{T_3 t_3^1 l_3}{E}$	.6866E-01	-.2932E-01	.6424E-02	-.1168E-02	.8051E-04	-.2996E-05	.3839E-07	-.3156E-08
64	Passive Elements	0.	0.	0.	0.	0.	0.	0.	0.
65	$C_1^* = \Sigma \frac{Q_i l_i}{A_i} (P)$	0.	0.	0.	0.	0.	0.	0.	0.
66	f_x^2	0.	0.	0.	0.	0.	0.	0.	0.
67	f_y^2	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
68	$K_2 \cdot f_y^2 - K_3 \cdot f_x^2$ (22)(67) - (23)(66)	0.	.5002E+06	.6481E+06	.8372E+06	.9114E+06	.9568E+06	.9783E+06	.9892E+06
69	$u_x^2 = 68/25$	0.	-.1829E-06	-.1675E-06	-.2821E-06	-.3618E-06	-.4235E-06	-.4592E-06	-.4790E-06
70	$K_2 \cdot f_x^2 - K_1 \cdot f_y^2$ (23)(66) - (21)(67)	-.1414E+06	-.1586E+07	-.1274E+07	-.1175E+07	-.1084E+07	-.1044E+07	-.1022E+07	-.1011E+07
71	$u_y^2 = 70/25$	.2929E-05	.3261E-06	.3291E-06	.3959E-06	.4302E-06	.4618E-06	.4796E-06	.4895E-06
72	$u_1^2 = (69)(9) + (71)(8)$	.2871E-05	.1578E-06	.1143E-06	.8046E-07	.4836E-07	.2715E-07	.1440E-07	.7429E-08
73	$u_2^2 = u_y^2$	.2929E-05	.3261E-06	.3291E-06	.3959E-06	.4302E-06	.4618E-06	.4796E-06	.4895E-06
74	$u_3^2 = -(69)(9) + (71)(8)$	.2871E-05	.3833E-06	.3511E-06	.4795E-06	.5600E-06	.6260E-06	.6639E-06	.6848E-06
75	$t_1^2 = \frac{A_1 u_1^2 E}{l_1} (16)(72)$	.2929E+00	.3292E+00	.2196E+00	.1619E+00	.9650E-01	.5430E-01	.2881E-01	.1486E-01
76	$t_2^2 = \frac{A_2 u_2^2 E}{l_2} (17)(73)$	.5858E+00	.5344E+00	.6894E+00	.7711E+00	.8635E+00	.9232E+00	.9593E+00	.9790E+00
77	$t_3^2 = \frac{A_3 u_3^2 E}{l_3} (18)(74)$	.2929E+00	.3292E+00	.2196E+00	.1619E+00	.9650E-01	.5430E-01	.2881E-01	.1486E-01
78	$Q_{12} = \frac{T_1 t_1^2 l_1}{E}$	.2678E+00	.3456E+00	.2166E+00	.1625E+00	.9646E-01	.5430E-01	.2881E-01	.1486E-01
79	$Q_{22} = \frac{T_2 t_2^2 l_2}{E}$	.3431E+00	.2406E+00	.3543E+00	.3827E+00	.4321E+00	.4616E+00	.4796E+00	.4895E+00
80	$Q_{32} = T_3 t_3^2 l_3 / E$	-.2513E-01	.1642E-01	-.3873E-02	.5937E-03	-.4013E-04	.1498E-05	-.1919E-07	.1578E-08

Problem 2(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
81	$C_2^* = \Sigma \frac{q_{i2}}{A_i}(P)$	0.	0.	0.	0.	0.	0.	0.	0.
82	$B_1 = \Sigma \frac{q_{i1}}{A_i}$ (Active)	.5000E-01	.4164E-01	.5000E-01	.4787E-01	.5000E-01	.4994E-01	.5000E-01	.5000E-01
83	$B_2 = \Sigma \frac{q_{i2}}{A_i}$ (Active)	.4142E-01	.5000E-01	.4582E-01	.5000E-01	.4952E-01	.5000E-01	.5000E-01	.5000E-01
84	$R_1 = 2(C_1^* + B_1 - \bar{C}_1) + B_1$	.5000E-01	.2491E-01	.5000E-01	.4361E-01	.5000E-01	.4981E-01	.5000E-01	.5000E-01
85	$R_2 = 2(C_2^* + B_2 - \bar{C}_2) + B_2$	.2426E-01	.5000E-01	.3747E-01	.5000E-01	.4856E-01	.5000E-01	.4999E-01	.5000E-01
86	$A = \Sigma \frac{q_{i1} q_{i1}}{\rho_i l_i A_i^3}$ (Active)	.2108E-04	.2680E-04	.4537E-04	.7128E-04	.9906E-04	.1209E-03	.1342E-03	.1417E-03
87	$B = \Sigma \frac{q_{i1} q_{i2}}{\rho_i l_i A_i^3}$ (Active)	.8579E-05	.4852E-05	.2795E-06	-.1306E-04	-.2463E-04	-.3554E-04	-.4209E-04	-.4586E-04
88	$C = \Sigma \frac{q_{i2} q_{i2}}{\rho_i l_i A_i^3}$ (Active)	.1194E-04	.1990E-04	.1996E-04	.3117E-04	.3665E-04	.4276E-04	.4604E-04	.4793E-04
89	$B \cdot B - A \cdot B$ (87)(87) - (86)(87)	-.1781E-09	-.5098E-09	-.9854E-09	-.2051E-08	-.3024E-08	-.3907E-08	-.4406E-08	-.4690E-08
90	$B \cdot R_2 - C \cdot R_1$ (87)(85) - (88)(84)	-.3890E-06	-.2531E-06	-.9874E-06	-.2013E-05	-.3029E-05	-.3907E-05	-.4406E-05	-.4690E-05
91	$\lambda_1 = 90/89$	.2184E+04	.4964E+03	.1091E+04	.9811E+03	.1002E+04	.9999E+03	.1000E+04	.1000E+04
92	$B \cdot R_1 - A \cdot R_2$ (87)(84) - (86)(85)	-.8253E-07	-.1219E-05	-.1686E-05	-.4134E-05	-.6042E-05	-.7815E-05	-.8813E-05	-.9380E-05
93	$\lambda_2 = 92/89$	.4633E+03	.2392E+04	.1862E+04	.2015E+04	.1998E+04	.2000E+04	.2000E+04	.2000E+04
94	IF (93) < 0 $\lambda_1 = R_1/A$	.2184E+04	.4964E+03	.1091E+04	.9811E+03	.1002E+04	.9999E+03	.1000E+04	.1000E+04
95	IF (93) < 0 $\lambda_2 = 0.0$	.4633E+03	.2392E+04	.1862E+04	.2015E+04	.1998E+04	.2000E+04	.2000E+04	.2000E+04
96	$\lambda_1 \cdot q_{11} + \lambda_2 q_{12}$ (94)(61) + (95)(78)	.1536E+04	.1257E+04	.1430E+04	.1406E+04	.1415E+04	.1414E+04	.1414E+04	.1414E+04
97	$\lambda_1 q_{21} + \lambda_2 q_{22}$ (94)(62) + (95)(79)	.1590E+03	.5377E+03	.4632E+03	.5836E+03	.4994E+03	.5000E+03	.5000E+03	.5000E+03
98	$\lambda_1 q_{31} + \lambda_2 q_{32}$ (94)(63) + (95)(80)	.1200E+03	.2471E+02	.1283E+01	.5006E-01	.4479E-03	.1499E-05	.7505E-09	.1750E-10
99	$96/(l_1 \cdot A_1^{*2} \cdot \rho)$	.1086E+01	.6773E+00	.9553E+00	.9455E+00	.9916E+00	.9987E+00	.1000E+01	.1000E+01
100	$97/(l_2 \cdot A_2^{*2} \cdot \rho)$	.1590E+00	.1327E+01	.7362E+00	.1022E+01	.9781E+00	.1000E+01	.9999E+00	.1000E+01

Problem 2(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
101	$98/(\ell_3 \cdot A_3^{*2} \cdot \rho)$	.8543E-01	.4916E-01	.8088E-02	.1195E-02	.4209E-04	.5631E-06	.1127E-08	.1052E-09
102	$A_1^{k+1} = A_1^{*}(1+0.5(\underline{99}-1))$	.1475E+02	.1359E+02	.1423E+02	.1411E+02	.1414E+02	.1414E+02	.1414E+02	.1414E+02
103	$A_2^{k+1} = A_2^{*}(1+0.5(100-1))$	.8195E+01	.1047E+02	.9738E+01	.1004E+02	.9995E+01	.1000E+02	.1000E+02	.1000E+02
104	$A_3^{k+1} = A_3^{*}(1+0.5(\underline{101}-1))$	.7675E+01	.4423E+01	.2388E+01	.1218E+01	.6134E+00	.3068E+00	.1534E+00	.7671E-01

Problem 2(b).

The design variables are modified by using the relation (Equation 158).

$$A_i^{k+1} = \frac{A_i^k}{\left(1 - \frac{1}{r}\left(\sum_{j=1}^{m} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} - 1\right)\right)}$$

with the step-size parameter $r = 2$. The Lagrange multipliers are determined by using the same equations as for problem 2(a) (See Equations 84 and 161). From row 48 it is seen that the weight of the structure after eight iterations is 156.0 lb, and a few more iterations are required in order to reach the minimum weight of 150.7 lb. The algorithm used for this problem is equivalent to the one derived by using the reciprocal design variable. This algorithm is generally slower to converge than the one used for problem 2(a).

Problem 2(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	A_1	.1000E+01	.1478E+02	.1487E+02	.1416E+02	.1414E+02	.1414E+02	.1414E+02	.1414E+02
2	A_2	.1000E+01	.9956E+01	.1112E+02	.9978E+01	.1001E+02	.9999E+01	.1000E+02	.1000E+02
3	A_3	.1000E+01	.9704E+01	.6363E+01	.4279E+01	.2857E+01	.1907E+01	.1271E+01	.8474E+00
4	$\overline{C_1} = \overline{C_2}$	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01	.5000E-01
5	E	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06
8	$\cos\theta$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.50[illegible]E+00	.5000E+00
12	$\cos\theta \cdot \cos\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.500[illegible]E+00	.5000E+00
13	l_1	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	l_2	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	l_3	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/l_1	.1414E+06	.2090E+07	.1990E+07	.2003E+07	.1999E+07	.2000E+07	.200[illegible]E+07	.2000E+07
17	A_2E/l_2	.2000E+06	.1991E+07	.2024E+07	.1994E+07	.2001E+07	.2000E+07	.200[illegible]E+07	.2000E+07
18	A_3E/l_3	.1414E+06	.1372E+07	.8999E+06	.6052E+06	.4041E+06	.2696E+06	.1798E+06	.1198E+06
19	$\frac{A_1E}{l_1} + \frac{A_3E}{l_3}$ (16+17)	.2828E+06	.3462E+07	.2890E+07	.2608E+07	.2403E+07	.2270E+07	.218[illegible]E+07	.2120E+07
20	$\frac{A_1E}{l_1} - \frac{A_3E}{l_3}$ (16-17)	0.	.7173E+06	.1090E+07	.1398E+07	.1595E+07	.1731E+07	.182[illegible]E+07	.1880E+07

Problem 2(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.1731E+07	.1445E+07	.1304E+07	.1202E+07	.1135E+07	.109[illegible]E+07	.1060E+07
22	$K_2 = (12)(20)$	0.	.3586E+06	.5452E+06	.6990E+06	.7976E+06	.8653E+06	.9101E+06	.9401E+06
23	$K_3 = (10)(19) +$	.3414E+06	.3722E+07	.3469E+07	.3298E+07	.3203E+07	.3135E+07	.3090E+07	.3060E+07
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.3005E+12	-.2378E+12	-.1900E+12	-.1608E+12	-.1404E+12	-.1270E+12	-.1180E+12
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4828E+11	-.6315E+13	-.4715E+13	-.3813E+13	-.3213E+13	-.2809E+13	-.2539E+13	-.2360E+13
26	$U_x = {}^{24}/_{25}$	.7071E+00	.4759E-01	.5044E-01	.4984E-01	.5004E-01	.4999E-01	.5000E-01	.5000E-01
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.2828E+11	-.3103E+12	-.2345E+12	-.1909E+12	-.1606E+12	-.1405E+12	-.1270E+12	-.1180E+12
28	$U_y = {}^{27}/_{25}$	.5858E+00	.4915E-01	.4973E-01	.5008E-01	.4998E-01	.5000E-01	.5000E-01	.5000E-01
29	$U_1 = (26)(9) + (28)(9)$	.9142E+00	.6840E-01	.7083E-01	.7065E-01	.7073E-01	.7071E-01	.7071E-01	.7071E-01
30	$U_2 = U_y\ (28)$	.5858E+00	.4915E-01	.4973E-01	.5008E-01	.4998E-01	.5000E-01	.5000E-01	.5000E-01
31	$U_3 = -(26)(9) + (28)(9)$	-.8579E-01	.1104E-02	-.4985E-03	.1682E-03	-.4717E-04	.1060E-04	-.1181E-05	.6806E-06
32	$T_1 = \frac{A_1 E}{l_1} U_1\ (16)(29)$	.1293E+06	.1429E+06	.1410E+06	.1415E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06
33	$T_2 = \frac{A_2 E}{l_2} U_2\ (17)(30)$	.1172E+06	.9786E+05	.1006E+06	.9986E+05	.1000E+06	.1000E+06	.1000E+06	.1000E+06
34	$T_3 = \frac{A_3 E}{l_3} U_3\ (18)(31)$	-.1213E+05	.1515E+04	-.4486E+03	.1018E+03	-.1986E+02	.2858E+01	-.2123E+00	.8157E-01
35	$\sigma_1 = \frac{T_1}{A_1} = {}^{32}/_{1}$	.1293E+06	.9673E+04	.1082E+05	.9991E+04	.1000E+05	.1000E+05	.1000E+05	.1000E+05
36	$\sigma_2 = {}^{T_2}/_{A_2} = {}^{33}/_{2}$	.1172E+06	.9829E+04	.9946E+04	.1002E+05	.9995E+04	.1000E+05	.1000E+05	.1000E+05
37	$\sigma_3 = {}^{T_3}/_{A_3} = {}^{34}/_{3}$	-.1213E+05	.1561E+03	-.7050E+02	.2379E+02	-.6671E+01	.1499E+01	-.167[illegible]E+00	.9625E-01
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\bar{\sigma}}$	.5172E+01	.3869E+00	.4007E+00	.3997E+00	.4001E+00	.4000E+00	.4000E+00	.4000E+00
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\bar{\sigma}}$	.4686E+01	.3932E+00	.3979E+00	.4006E+00	.3998E+00	.4000E+00	.4000E+00	.4000E+00
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\bar{\sigma}}$	-.4853E+00	.6244E-02	-.2820E-02	.9516E-03	-.2669E-03	.5995E-04	-.668[illegible]E-05	.3850E-05

Problem 2(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(U_x) = \frac{U_x}{\varepsilon_1} = \frac{26}{4}$	.1414E+02	.9517E+00	.1009E+01	.9968E+00	.1001E+01	.9998E+00	.1000E+01	.1000E+01
42	$\Lambda(U_y) = \frac{U_y}{\varepsilon_1} = \frac{28}{4}$	.1172E+02	.9829E+00	.9946E+00	.1002E+01	.9995E+00	.1000E+01	.100 E+01	.1000E+01
43	$\Lambda(max)$ (38-42)	.1414E+02	.9829E+00	.1009E+01	.1002E+01	.1001E+01	.1000E+01	.100 E+01	.1000E+01
44	$A_1^* = \Lambda \cdot A_1$ (43)(1)	.1414E+02	.1452E+02	.1420E+02	.1419E+02	.1415E+02	.1414E+02	.1414E+02	.1414E+02
45	$A_2^* = \Lambda \cdot A_2$ (43)(2)	.1414E+02	.9786E+01	.1021E+02	.9986E+01	.1002E+02	.1000E+02	.1000E+02	.1000E+02
46	$A_3^* = \Lambda \cdot A_3$ (43)(3)	.1414E+02	.9539E+01	.6419E+01	.4286E+01	.2860E+01	.1907E+01	.1271E+01	.8474E+00
47	VOLUME $= \Sigma A_i l_i$	.2707E+04	.2191E+04	.1968E+04	.1805E+04	.1704E+04	.1635E+04	.1590E+04	.1560E+04
48	WT = (47)(ρ)	.2707E+03	.2191E+03	.1968E+03	.1805E+03	.1704E+03	.1635E+03	.1590E+03	.1560E+03
49	f_x'	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
50	f_y'	0.	0.	0.	0.	0.	0.	0.	0.
51	$K_2 \cdot f_y - K_3 \cdot f_x$ (22)(50)-(23)(49)	-.3414E+06	-.3722E+07	-.3469E+07	-.3298E+07	-.3203E+07	-.3135E+07	-.3090E+07	-.3060E+07
52	$u_x' = 51/25$	.7071E-05	.5895E-06	.7356E-06	.8650E-06	.9969E-06	.1116E-05	.1217E-05	.1297E-05
53	$K_2 \cdot f_x - K_1 \cdot f_y$ (22)(49)-(21)(50)	0.	.3586E+06	.5452E+06	.6990E+06	.7976E+06	.8653E+06	.9161E+06	.9401E+06
54	$u_y' = 53/25$	0.	-.5679E-07	-.1156E-06	-.1833E-06	-.2482E-06	-.3081E-06	-.3584E-06	-.3984E-06
55	$u_1' = (52)(9)+(54)(8)$	.5000E-05	.3766E-06	.4384E-06	.4820E-06	.5294E-06	.5713E-06	.607 E-06	.6353E-06
56	$u_2' = u_y$ (54)	0.	-.5679E-07	-.1156E-06	-.1833E-06	-.2482E-06	-.3081E-06	-.3584E-06	-.3984E-06
57	$u_3' = -(52)(9)+(54)(8)$	-.5000E-05	-.4570E-06	-.6019E-06	-.7413E-06	-.8805E-06	-.1007E-05	-.1114E-05	-.1199E-05
58	$t_1' = \frac{A_1 u_1' E}{l_1}$ (16)(55)	.7071E+00	.7071E+00	.8726E+00	.9656E+00	.1050E+01	.1143E+01	.1214E+01	.1271E+01
59	$t_2' = \frac{A_2 u_2' E}{l_2}$ (17)(56)	0.	-.1131E+00	-.2340E+00	-.3655E+00	-.4960E+00	-.6160E+00	-.7168E+00	-.7968E+00
60	$t_3' = \frac{A_3 u_3' E}{l_3}$ (18)(57)	-.7071E+00	-.6271E+00	-.5417E+00	-.4486E+00	-.3558E+00	-.2715E+00	-.2062E+00	-.1437E+00

Problem 2(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$Q_{11} = \frac{T_1 t_1^1 l_1}{E}$	.6465E+00	.7955E+00	.8690E+00	.9663E+00	.1058E+01	.1143E+01	.1214E+01	.1271E+01
62	$Q_{21} = \frac{T_2 t_2^1 l_2}{E}$	0.	-.5533E-01	-.1177E+00	-.1825E+00	-.2485E+00	-.3080E+00	-.3584E+00	-.3984E+00
63	$Q_{31} = \frac{T_3 t_3^1 l_3}{E}$	.6066E-01	-.6718E-02	.1718E-02	-.3238E-03	.4796E-04	-.5486E-05	.3006E-06	-.8286E-07
64	Passive Elements	0.	0.	0.	0.	0.	0.	0.	0.
65	$C_1^* = \Sigma \frac{Q \lambda_1}{A \lambda} (P)$	0.	0.	0.	0.	0.	0.	0.	0.
66	f_x^2	0.	0.	0.	0.	0.	0.	0.	0.
67	f_y^2	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.100CE+01	.1000E+01
68	$K_2 \cdot f_y^2 - K_3 \cdot f_x^2$ (22)(67) - (23)(66)	0.	.3586E+06	.5452E+06	.6998E+06	.7976E+06	.8653E+06	.9101E+06	.9401E+06
69	$u_x^2 = 68/25$	0.	-.5679E-07	-.1156E-06	-.1833E-06	-.2482E-06	-.3081E-06	-.3584E-06	-.3984E-06
70	$K_2 \cdot f_x^2 - K_1 \cdot f_y^2$ (23)(66) - (21)(67)	-.1414E+06	-.1731E+07	-.1445E+07	-.1384E+07	-.1202E+07	-.1135E+07	-.1093E+07	-.1060E+07
71	$u_y^2 = 70/25$	.2929E-05	.2741E-06	.3065E-06	.3420E-06	.3740E-06	.4041E-06	.4292E-06	.4492E-06
72	$u_1^2 = (69)(9) + (71)(8)$	.2871E-05	.1537E-06	.1349E-06	.1122E-06	.8893E-07	.6788E-07	.5006E-07	.3592E-07
73	$u_2^2 = u_y^2$	.2929E-05	.2741E-06	.3065E-06	.3420E-06	.3740E-06	.4041E-06	.4292E-06	.4492E-06
74	$u_3^2 = -(69)(9) + (71)(8)$	.2871E-05	.2340E-06	.2985E-06	.3715E-06	.4400E-06	.5035E-06	.5569E-06	.5994E-06
75	$t_1^2 = \frac{A_1 u_1^2 E}{l_1}$ (16)(72)	.2929E+00	.3211E+00	.2686E+00	.2248E+00	.1778E+00	.1358E+00	.1001E+00	.7183E-01
76	$t_2^2 = \frac{A_2 u_2^2 E}{l_2}$ (17)(73)	.5858E+00	.5458E+00	.6202E+00	.6821E+00	.7486E+00	.8080E+00	.8584E+00	.8984E+00
77	$t_3^2 = \frac{A_3 u_3^2 E}{l_3}$ (18)(74)	.2929E+00	.3211E+00	.2686E+00	.2248E+00	.1778E+00	.1358E+00	.1001E+00	.7183E-01
78	$Q_{12} = \frac{T_1 t_1^2 l_1}{E}$	.2678E+00	.3246E+00	.2677E+00	.2258E+00	.1778E+00	.1358E+00	.1001E+00	.7183E-01
79	$Q_{22} = \frac{T_2 t_2^2 l_2}{E}$	.3431E+00	.2671E+00	.3120E+00	.3405E+00	.3744E+00	.4040E+00	.4292E+00	.4492E+00
80	$Q_{32} = T_3 t_3^2 l_3 / E$	-.2513E-01	.3440E-02	-.8520E-03	.1619E-03	-.2396E-04	.2743E-05	-.1503E-06	.4143E-07

Problem 2(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
81	$C_2^* = \Sigma \frac{q_{i2}}{A_i}(P)$	0.	0.	0.	0.	0.	0.	0.	0.
82	$B_1 = \Sigma \frac{q_{i1}}{A_i}$ (Active)	.5000E-01	.4841E-01	.5000E-01	.4976E-01	.5000E-01	.4999E-01	.5000E-01	.5000E-01
83	$B_2 = \Sigma \frac{q_{i2}}{A_i}$ (Active)	.4142E-01	.5000E-01	.4930E-01	.5000E-01	.4993E-01	.5000E-01	.5000E-01	.5000E-01
84	$R_1 = 2(c_1^* + B_1 - \bar{c}_1) + B_1$	.5000E-01	.4524E-01	.5000E-01	.4929E-01	.5000E-01	.4996E-01	.5000E-01	.5000E-01
85	$R_2 = 2(c_2^* + B_2 - \bar{c}_2) + B_2$	.2426E-01	.5000E-01	.4790E-01	.5000E-01	.4980E-01	.5000E-01	.5000E-01	.5000E-01
86	$A = \Sigma \frac{q_{i1} q_{i1}}{P_i \lambda_i A_i^3}$ (Active)	.2100E-04	.2987E-04	.4000E-04	.5295E-04	.6820E-04	.8424E-04	.9937E-04	.1125E-03
87	$B = \Sigma \frac{q_{i1} q_{i2}}{P_i \lambda_i A_i^3}$ (Active)	.8579E-05	.8761E-05	.4590E-05	-.1715E-05	-.9125E-05	-.1713E-04	-.2469E-04	-.3123E-04
88	$C = \Sigma \frac{q_{i2} q_{i2}}{P_i \lambda_i A_i^3}$ (Active)	.1194E-04	.2009E-04	.2186E-04	.2580E-04	.2940E-04	.3357E-04	.3734E-04	.4062E-04
89	$B \cdot B - A \cdot B$ (87)(87) - (86)(89)	-.1701E-09	-.5233E-09	-.8534E-09	-.1363E-08	-.1927E-08	-.2534E-08	-.3101E-08	-.3592E-08
90	$B \cdot R_2 - C \cdot R_1$ (87)(85) - (88)(84)	-.3090E-06	-.4707E-06	-.8729E-06	-.1357E-05	-.1928E-05	-.2534E-05	-.3101E-05	-.3592E-05
91	$\lambda_1 = 90/89$	.2104E+04	.8995E+03	.1023E+04	.9958E+03	.1001E+04	.9999E+03	.1000E+04	.1000E+04
92	$B \cdot R_1 - A \cdot R_2$ (87)(84) - (86)(85)	-.0253E-07	-.1097E-05	-.1686E-05	-.2732E-05	-.3853E-05	-.5068E-05	-.6203E-05	-.7185E-05
93	$\lambda_2 = 92/89$	.4633E+03	.2097E+04	.1976E+04	.2004E+04	.1999E+04	.2000E+04	.2000E+04	.2000E+04
94	IF (93) < 0 $\lambda_1 = R_1/A$	.2104E+04	.8995E+03	.1023E+04	.9958E+03	.1001E+04	.9999E+03	.1000E+04	.1000E+04
95	IF (93) < 0 $\lambda_2 = 0.0$	.4633E+03	.2097E+04	.1976E+04	.2004E+04	.1999E+04	.2000E+04	.2000E+04	.2000E+04
96	$\lambda_1 q_{11} + \lambda_2 q_{12}$ (94)(61) + (95)(78)	.1536E+04	.1396E+04	.1419E+04	.1413E+04	.1414E+04	.1414E+04	.1414E+04	.1414E+04
97	$\lambda_1 q_{21} + \lambda_2 q_{22}$ (94)(62) + (95)(79)	.1598E+03	.5102E+03	.4962E+03	.5007E+03	.4999E+03	.5000E+03	.5000E+03	.5000E+03
98	$\lambda_1 q_{31} + \lambda_2 q_{32}$ (94)(63) + (95)(80)	.120[illegible]+03	.1171E+01	.7377E-01	.2740E-02	.7266E-04	.1278E-05	.5766E-08	.7242E-09
99	$96/(\lambda_1 \cdot A_1^{*2} \cdot P)$	.1006E+01	.9360E+00	.9955E+00	.9931E+00	.9991E+00	.9997E+00	.1000E+01	.1000E+01
100	$97/(\lambda_2 \cdot A_2^{*2} \cdot P)$	.1590E+00	.1066E+01	.9527E+00	.1004E+01	.9965E+00	.1000E+01	.9999E+00	.1000E+01

Problem 2(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
101	$98/(l_3 \cdot A_3^{*2} \cdot P)$	.8543E-01	.1820E-02	.2532E-03	.2110E-04	.1256E-05	.4973E-07	.5046E-09	.1426E-09
102	$A_1^{k+1} = A_1^{*}/(1-0.5(\underline{99}-1))$	.1470E+02	.1407E+02	.1416E+02	.1414E+02	.1414E+02	.1414E+02	.1414E+02	.1414E+02
103	$A_2^{k+1} = A_2^{*}/(1-0.5(\underline{100}-1))$	.9956E+01	.1012E+02	.9970E+01	.1001E+02	.9999E+01	.1000E+02	.1000E+02	.1000E+02
104	$A_3^{k+1} = A_3^{*}/(1-0.5(\underline{101}-1))$	.9704E+01	.6363E+01	.4279E+01	.2857E+01	.1907E+01	.1271E+01	.8474E+00	.5650E+00

Problem 2(c).

In this problem the relative design vector α_i (See Equation 17) is used. This is modified by using the relation

$$\alpha_i^{k+1} = \alpha_i^k \left(\frac{Q_{i1}}{\alpha_i^2 \rho_i \ell_i} \right)_k^{1/2}$$

and the Lagrange multipliers are estimated from the relation (Equation 74)

$$\lambda_j^{k+1} = \lambda_j^k \left(\frac{C_j}{\bar{C}_j} \right)_k^{1/2} \qquad j = 1,2$$

For the first iteration $\lambda_1 = \lambda_2 = 1.0$. The Lagrange multipliers and the relative design variables α_i are normalized after each modification so that the maximum α_i or λ_j is equal to unity. A single virtual-load vector with $f_x = \lambda_1 \cdot 1$ and $f_y = \lambda_2 . 1$ is used.

After eight iterations the weight of the structure is 150.8 lb (See Row No. 48), but A_3 for this design is 0.0999 in^2, which should be equal to 0.1 in^2. The minimum-weight design is nearly the same as the one for Problem 2(a).

Problem 2(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	α_1	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
2	α_2	.1000E+01	.7206E+00	.3693E+00	.9169E+00	.8486E+00	.7267E+00	.7065E+00	.7067E+00
3	α_3	.1000E+01	.1971E+00	.2317E-01	.3232E-01	.5070E-02	.6079E-02	.6090E-02	.7065E-02
4		~~.5000E-01~~	~~.5000E-01~~	~~.5000E-01~~	~~.5000E-01~~	~~.5000E-01~~	~~.5000E-01~~	~~.5000E-01~~	~~.5000E-01~~
5	E	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06
8	$\cos\theta$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta \cdot \cos\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	l_1	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	l_2	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	l_3	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/l_1	.1414E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06
17	A_2E/l_2	.2000E+06	.1457E+06	.7386E+05	.1834E+06	.1697E+06	.1453E+06	.1413E+06	.1413E+06
18	A_3E/l_3	.1414E+06	.2788E+05	.3277E+04	.4571E+04	.8302E+03	.8597E+03	.9744E+03	.9992E+03
19	$\frac{A_1E}{l_1} + \frac{A_3E}{l_3}$ (16+17)	.2828E+06	.1693E+06	.1447E+06	.1460E+06	.1423E+06	.1423E+06	.1424E+06	.1424E+06
20	$\frac{A_1E}{l_1} - \frac{A_3E}{l_3}$ (16−17)	0.	.1135E+06	.1381E+06	.1369E+06	.1406E+06	.1406E+06	.1404E+06	.1404E+06

Problem 2(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.8465E+05	.7235E+05	.7300E+05	.7113E+05	.7114E+05	.7120E+05	.7121E+05
22	$K_2 = (12)(20)$	0.	.5677E+05	.6987E+05	.6043E+05	.7030E+05	.7020E+05	.7022E+05	.7021E+05
23	$K_3 = (10)(19) + 17$	.3414E+06	.2384E+06	.1462E+06	.2564E+06	.2408E+06	.2165E+06	.2125E+06	.2126E+06
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.1160E+11	-.8060E+09	-.1195E+11	-.1002E+11	-.7592E+10	-.7285E+10	-.7213E+10
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4020E+11	-.1620E+11	-.5007E+10	-.1403E+11	-.1219E+11	-.1046E+11	-.1020E+11	-.1021E+11
26	$U_x = {}^{24}/_{25}$	.7071E+00	.7177E+00	.1309E+00	.8510E+00	.8225E+00	.7257E+00	.7065E+00	.7067E+00
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.2020E+11	-.1125E+11	-.7563E+10	-.7757E+10	-.7196E+10	-.7200E+10	-.7217E+10	-.7221E+10
28	$U_y = {}^{27}/_{25}$	.5050E+00	.6913E+00	.1302E+01	.5520E+00	.5904E+00	.6003E+00	.7077E+00	.7075E+00
29	$U_1 = (26)(9) + (28)(8)$	.9142E+00	.9963E+00	.1019E+01	.9932E+00	.9990E+00	.9998E+00	.1000E+01	.1000E+01
30	$U_2 = U_y\ (28)$	.5050E+00	.6913E+00	.1302E+01	.5520E+00	.5904E+00	.6003E+00	.7077E+00	.7075E+00
31	$U_3 = -(26)(9) + (28)(8)$	-.0579E-01	-.1064E-01	.0226E+00	-.2114E+00	-.1641E+00	-.2647E-01	.0225E-03	.5102E-03
32	$T_1 = \frac{A_1 E}{l_1} U_1\ (16)(29)$	.1293E+06	.1409E+06	.1441E+06	.1405E+06	.1413E+06	.1414E+06	.1414E+06	.1414E+06
33	$T_2 = \frac{A_2 E}{l_2} U_2\ (17)(30)$	.1172E+06	.1007E+06	.9619E+05	.1014E+06	.1002E+06	.1000E+06	.1000E+06	.1000E+06
34	$T_3 = \frac{A_3 E}{l_3} U_3\ (18)(31)$	-.1213E+05	-.5198E+03	.2695E+04	-.9663E+03	-.1363E+03	-.2276E+02	.0014E+00	.5177E+00
35	$\sigma_1 = \frac{T_1}{A_1} = {}^{32}/_{1}$	.1293E+06	.1409E+06	.1441E+06	.1405E+06	.1413E+06	.1414E+06	.1414E+06	.1414E+06
36	$\sigma_2 = {}^{T_2}/_{A_2} = {}^{33}/_{2}$	.1172E+06	.1303E+06	.2604E+06	.1106E+06	.1101E+06	.1377E+06	.1415E+06	.1415E+06
37	$\sigma_3 = {}^{T_3}/_{A_3} = {}^{34}/_{3}$	-.1213E+05	-.2636E+04	.1163E+06	-.2990E+05	-.2321E+05	-.3744E+04	.1163E+03	.7328E+02
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\sigma_1'}$ (Ab)	.5172E+01	.5636E+01	.5765E+01	.5618E+01	.5651E+01	.5656E+01	.5657E+01	.5657E+01
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\sigma_2'}$ (Ab)	.4686E+01	.5531E+01	.1042E+02	.4422E+01	.4723E+01	.5506E+01	.5662E+01	.5660E+01
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\sigma_3'}$ (Ab)	.4053E+00	.1055E+00	.4653E+01	.1196E+01	.9285E+00	.1498E+00	.4653E-02	.2931E-02

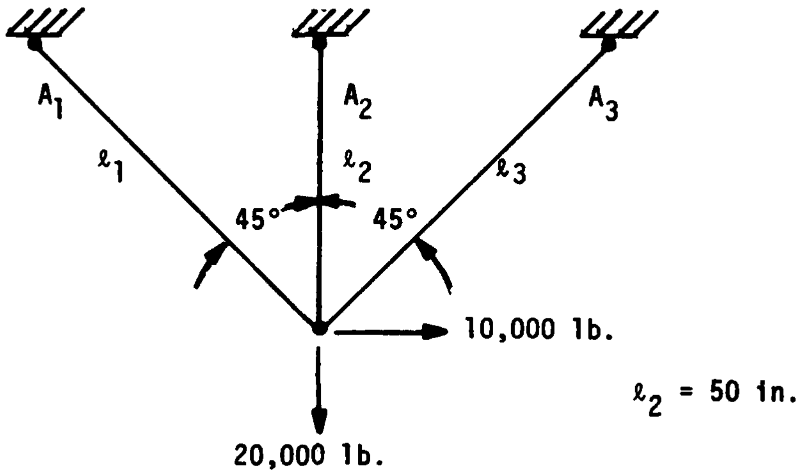
Problem 2(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(U_x) = \frac{U_x}{\bar{\varepsilon}_1} = \frac{26}{4}$	.1414E+02	.1435E+02	.2778E+01	.1704E+02	.1645E+02	.1451E+02	.1413E+02	.1413E+02
42	$\Lambda(U_y) = \frac{U_y}{\bar{\varepsilon}_1} = \frac{28}{4}$	.1172E+02	.1383E+02	.2604E+02	.1106E+02	.1101E+02	.1377E+02	.1415E+02	.1415E+02
43	$\Delta(\max)$ (38–42)	.1414E+02	.1435E+02	.2604E+02	.1704E+02	.1645E+02	.1451E+02	.1415E+02	.1415E+02
44	$A_1^* = \Lambda \cdot A_1$ (43)(1)	.1414E+02	.1435E+02	.2604E+02	.1704E+02	.1645E+02	.1451E+02	.1415E+02	.1415E+02
45	$A_2^* = \Lambda \cdot A_2$ (43)(2)	.1414E+02	.1046E+02	.9619E+01	.1562E+02	.1396E+02	.1055E+02	.1000E+02	.1000E+02
46	$A_3^* = \Lambda \cdot A_3$ (43)(3)	.1414E+02	.2030E+01	.6034E+00	.9596E+00	.9656E-01	.8823E-01	.9752E-01	.9997E-01
47	VOLUME = $\Sigma A_i l_i$	.2707E+04	.1730E+04	.2365E+04	.2024E+04	.1060E+04	.1560E+04	.1500E+04	.1500E+04
48	WT = (47)(ρ)	.2707E+03	.1730E+03	.2365E+03	.2024E+03	.1060E+03	.1560E+03	.1500E+03	.1500E+03
49	$\lambda_1^{k+1} = \lambda_1^k (41)^{1/2}$	.1000E+01	.3709E+01	.1667E+01	.1374E+01	.1075E+01	.1050E+01	.1002E+01	.1001E+01
50	$\lambda_2^{k+1} = \lambda_2^k (42)^{1/2}$	.1000E+01	.3710E+01	.5009E+01	.3325E+01	.3436E+01	.3710E+01	.3762E+01	.3762E+01
51	$\lambda_1^{k+1} = 49/\mathrm{Max}(49,50)$	.1000E+01	.1000E+01	.3320E+00	.4131E+00	.4076E+00	.5007E+00	.5003E+00	.5000E+00
52	$\lambda_2^{k+1} = 50/\mathrm{Max}(49,50)$	.1000E+01	.9815E+00	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
53	$f_x' = \lambda_1^{k+1} \cdot 1$ (51·1)	.1000E+01	.1000E+01	.3320E+00	.4131E+00	.4076E+00	.5007E+00	.5003E+00	.5000E+00
54	$f_y' = \lambda_2^{k+1} \cdot 1$ (52·1)	.1000E+01	.9815E+00	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
55	$K_2 \cdot f_y' - K_3 \cdot f_x'$ (22)(54)–(23)(53)	-.3414E+06	-.1746E+06	.2042E+05	-.3740E+05	-.4713E+05	-.3810E+05	-.3600E+05	-.3607E+05
56	$U_x = 55/25$	.7071E-05	.1073E-04	-.3515E-05	.2671E-05	.3067E-05	.3642E-05	.3530E-05	.3534E-05
57	$K_2 \cdot f_x' - K_1 \cdot f_y'$ (22)(53)–(21)(54)	-.1414E+06	-.2631E+05	-.4936E+05	-.4473E+05	-.3605E+05	-.3595E+05	-.3607E+05	-.3611E+05
58	$U_y = 57/25$	.2929E-05	.1616E-05	.8500E-05	.3100E-05	.3023E-05	.3437E-05	.3537E-05	.3537E-05
59	$u_1' = (56)(9) + (58)(8)$	.7071E-05	.8730E-05	.3525E-05	.4143E-05	.4872E-05	.5006E-05	.5003E-05	.5000E-05
60	$u_2' = u_y$	.2929E-05	.1616E-05	.8500E-05	.3100E-05	.3023E-05	.3437E-05	.3537E-05	.3537E-05

Problem 2(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$u_3' = -(56)(9)+(57)(8)$	-.2929E-05	-.6444E-05	.0496E-05	.3654E-06	-.5965E-06	-.1453E-06	-.9100E-09	.2640E-08
62	$t_1' = \frac{A_1 u_1' E}{l_1}(16)(59)$	.1000E+01	.1235E+01	.4985E+00	.5859E+00	.6891E+00	.7079E+00	.7075E+00	.7071E+00
63	$t_2' = \frac{A_2 u_2' E}{l_2}(17)(60)$	.5050E+00	.2355E+00	.6278E+00	.5845E+00	.5131E+00	.4995E+00	.4997E+00	.5000E+00
64	$t_3' = \frac{A_3 u_3' E}{l_3}(18)(61)$	-.4142E+00	-.1797E+00	.2784E-01	.1670E-02	-.4952E-03	-.1249E-03	-.0952E-06	.2638E-05
65	$Q_{11} = \frac{T_1 t_1' l_1}{E}$	.9142E+00	.1230E+01	.5000E+00	.5819E+00	.6004E+00	.7070E+00	.7075E+00	.7071E+00
66	$Q_{21} = \frac{T_2 t_2' l_2}{E}$	.3431E+00	.1186E+00	.3020E+00	.2963E+00	.2571E+00	.2498E+00	.2499E+00	.2500E+00
67	$Q_{31} = \frac{T_3 t_3' l_3}{E}$	.3553E-01	.6603E-03	.5306E-03	-.1141E-04	.4771E-06	.2011E-07	-.5073E-11	.9656E-11
68	$Q_{11} / \alpha_1^2 \cdot \rho \cdot l_1$	.1293E+00	.1740E+00	.7184E-01	.8229E-01	.9735E-01	.1001E+00	.1001E+00	.1000E+00
69	$Q_{21} / \alpha_2^2 \cdot \rho \cdot l_2$	.6063E-01	.4470E-01	.4420E+00	.7048E-01	.7140E-01	.9462E-01	.1001E+00	.1001E+00
70	$Q_{31} / \alpha_3^2 \cdot \rho \cdot l_3$	.5029E-02	.2403E-02	.1398E+00	-.1545E-02	.1958E-02	.7694E-04	-.1511E-07	.2736E-07
71	$\alpha_1^{k+1} = \alpha_1 (68)^{1/2}$	.3596E+00	.4171E+00	.2600E+00	.2869E+00	.3120E+00	.3164E+00	.3163E+00	.3162E+00
72	$\alpha_2^{k+1} = \alpha_2 (69)^{1/2}$	.2620E+00	.1540E+00	.2457E+00	.2434E+00	.2267E+00	.2235E+00	.2235E+00	.2236E+00
73	$\alpha_3^{k+1} = \alpha_3 (70)^{1/2}$	.7089E-01	.9663E-02	.0662E-02	0.	.2598E-03	.5332E-04	0.	.1169E-05
74	$\alpha_1 = 71/\mathrm{Max}(71-73)$	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
75	$\alpha_2 = 72/\mathrm{Max}(71-73)$	.7206E+00	.3693E+00	.9169E+00	.8486E+00	.7267E+00	.7065E+00	.7067E+00	.7071E+00
76	$\alpha_3 = 73/\mathrm{Max}(71-73)$	.1971E+00	.2317E-01	.3232E-01	0.	.8325E-03	.1685E-03	0.	.3695E-05
77	α_1 checked ~~for A min~~	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
78	α_2 $\alpha_{min} = \frac{A_{min}}{\Lambda(max)}$	.7206E+00	.3693E+00	.9169E+00	.8486E+00	.7267E+00	.7065E+00	.7067E+00	.7071E+00
79	α_3 (43)	.1971E+00	.2317E-01	.3232E-01	.5070E-02	.6079E-02	.6090E-02	.7065E-02	.7067E-02

Problem 3. Stress Constrained Problem.



$E = 10^7$ psi $\qquad \rho = 0.1$ lb/in^3 $\qquad A_{min} = 0.1$ in^2

$\bar{\sigma}_1 = 10{,}000$ psi $\qquad \bar{\sigma}_2 = 40{,}000$ psi $\qquad \bar{\sigma}_3 = 10{,}000$ psi

This problem is solved by using three different algorithms. First the structure is designed by using the F.S.D. method, then the other two approaches are used. For this problem the F.S.D. algorithm does not lead to the real minimum-weight design.

Problem 3(a).

The design variables are modified by (Equation 117)

$$A_i^{k+1} = \left(\frac{T_i}{\sigma_i}\right)_k$$

This is the Fully Stressed Design algorithm. The weight of the structure with the same size for all the elements is 247.5 lb. After the first iteration the weight reduced to 158.4 lb. In the second iteration the weight increased to 246.7 lb and then decreased with subsequent iterations until it reached 198.5 lb. The cross-sectional areas for the design with a weight equal to 198.5 lb are A_1 = 21.07 in^2, A_2 = 0.1 in^2 and A_3 = 6.92 in^2. The stress in the second element is equal to 20,000 psi; in the other two elements, 10,000 psi. The size of A_2 is dictated by the minimum-size requirement. This is not an optimum design (See Problem 3(b) or 3(c)).

Problem 3(a) Cont'd.

			CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	A_1		.1000E+01	.1293E+02	.1502E+02	.1725E+02	.1869E+02	.1976E+02	.2043E+02	.2081E+02
2	A_2		.1000E+01	.2929E+01	.1907E+01	.1403E+01	.8935E+00	.5126E+00	.2759E+00	.1433E+00
3	A_3		.1000E+01	.1213E+01	.1676E+01	.3103E+01	.4544E+01	.5621E+01	.6291E+01	.6666E+01
4			.1000E+09	.1000E+09	.1000E+09	.1000E+09	.1000E+09	.1000E+09	.1000E+09	.1000E+09
5	E		.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x		.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y		.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06
8	$\cos\theta$		.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$		.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$		.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$		.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta \cdot \cos\phi$		.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	ℓ_1		.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	ℓ_2		.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	ℓ_3		.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/ℓ_1		.1414E+06	.1828E+07	.2237E+07	.2439E+07	.2643E+07	.2795E+07	.2890E+07	.2943E+07
17	A_2E/ℓ_2		.2000E+06	.5858E+06	.3815E+06	.2806E+06	.1787E+06	.1025E+06	.5518E+05	.2866E+05
18	A_3E/ℓ_3		.1414E+06	.1716E+06	.2370E+06	.4389E+06	.6426E+06	.7950E+06	.8897E+06	.9427E+06
19	$\frac{A_1E}{\ell_1} + \frac{A_3E}{\ell_3}$ (16+17)		.2828E+06	.2000E+07	.2474E+07	.2878E+07	.3285E+07	.3590E+07	.3779E+07	.3885E+07
20	$\frac{A_1E}{\ell_1} - \frac{A_3E}{\ell_3}$ (16−17)		0.	.1657E+07	.2000E+07	.2000E+07	.2000E+07	.2000E+07	.2000E+07	.2000E+07

Problem 3(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.1000E+07	.1237E+07	.1439E+07	.1643E+07	.1795E+07	.1890E+07	.1943E+07
22	$K_2 = (12)(20)$	0.	.0284E+06	.1000E+07	.1000E+07	.1000E+07	.1000E+07	.1000E+07	.1000E+07
23	$K_3 = (10)(19) + 17$	.3414E+06	.1506E+07	.1619E+07	.1719E+07	.1821E+07	.1897E+07	.1945E+07	.1971E+07
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	.7180E+10	.3815E+11	.2806E+11	.1787E+11	.1025E+11	.5518E+10	.2066E+10
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4020E+11	-.8995E+12	-.1002E+13	-.1474E+13	-.1992E+13	-.2406E+13	-.2675E+13	-.2038E+13
26	$U_x = {}^{24}/_{25}$	.7071E+00	-.7902E-02	-.3807E-01	-.1903E-01	-.8972E-02	-.4261E-02	-.2063E-02	-.1013E-02
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.2020E+11	-.1172E+12	-.1474E+12	-.1878E+12	-.2285E+12	-.2590E+12	-.2779E+12	-.2885E+12
28	$U_y = {}^{27}/_{25}$	.5050E+00	.1302E+00	.1471E+00	.1274E+00	.1147E+00	.1076E+00	.1039E+00	.1020E+00
29	$U_1 = (26)(9) + (28)(9)$	.9142E+00	.8651E-01	.7709E-01	.7662E-01	.7479E-01	.7311E-01	.7201E-01	.7139E-01
30	$U_2 = U_y\ (28)$	.5050E+00	.1302E+00	.1471E+00	.1274E+00	.1147E+00	.1076E+00	.1039E+00	.1020E+00
31	$U_3 = -(26)(9) + (28)(9)$	-.8579E-01	.9769E-01	.1309E+00	.1035E+00	.8740E-01	.7913E-01	.7493E-01	.7282E-01
32	$T_1 = \frac{A_1 E}{l_1} U_1\ (16)(29)$	.1293E+06	.1502E+06	.1725E+06	.1869E+06	.1976E+06	.2043E+06	.2081E+06	.2101E+06
33	$T_2 = \frac{A_2 E}{l_2} U_2\ (17)(30)$	.1172E+06	.7630E+05	.5611E+05	.3574E+05	.2050E+05	.1104E+05	.5733E+04	.2923E+04
34	$T_3 = \frac{A_3 E}{l_3} U_3\ (18)(31)$	-.1213E+05	.1676E+05	.3103E+05	.4544E+05	.5621E+05	.6291E+05	.6666E+05	.6864E+05
35	$\sigma_1 = \frac{T_1}{A_1} = {}^{32}/_1$	.1293E+06	.1223E+05	.1090E+05	.1084E+05	.1050E+05	.1034E+05	.1018E+05	.1010E+05
36	$\sigma_2 = {}^{T_2}/_{A_2} = {}^{33}/_2$	.1172E+06	.2605E+05	.2942E+0[illegible]	.2540E+05	.2295E+05	.2153E+05	.2078E+05	.2039E+05
37	$\sigma_3 = {}^{T_3}/_{A_3} = {}^{34}/_3$	-.1213E+05	.1382E+05	.1052E+05	.1464E+05	.1237E+05	.1119E+05	.1060E+05	.1030E+05
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\bar{\sigma}}\ (Abs)$	.1293E+02	.1223E+01	.1090E+01	.1084E+01	.1050E+01	.1034E+01	.1018E+01	.1010E+01
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\bar{\sigma}}\ (Abs)$	.2929E+01	.6512E+00	.7354E+00	.6369E+00	.5737E+00	.5382E+00	.5195E+00	.5098E+00
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\bar{\sigma}}\ (Abs)$	.1213E+01	.1382E+01	.1052E+01	.1464E+01	.1237E+01	.1119E+01	.1060E+01	.1030E+01

Problem 3(a) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41		~~.7871E-08~~	~~.7903E-10~~	~~.3887E-09~~	~~.1903E-09~~	~~.8978E-10~~	~~.4261E-10~~	~~.2063E-10~~	~~.1018E-11~~
42		~~.5850E-08~~	~~.1302E-08~~	~~.1471E-08~~	~~.1274E-08~~	~~.1147E-08~~	~~.1076E-08~~	~~.1039E-08~~	~~.1020E-08~~
43	$\Delta(max)_{(38-42)}$	.1293E+02	.1382E+01	.1852E+01	.1464E+01	.1237E+01	.1119E+01	.1060E+01	.1030E+01
44	$A_1^* = \Lambda \cdot A_{1(43)(1)}$	.1293E+02	.1786E+02	.2929E+02	.2525E+02	.2312E+02	.2212E+02	.2165E+02	.2143E+02
45	$A_2^* = \Lambda \cdot A_{2(43)(2)}$	.1293E+02	.4046E+01	.3532E+01	.2054E+01	.1185E+01	.5736E+00	.2923E+00	.1476E+00
46	$A_3^* = \Lambda \cdot A_{3(43)(3)}$	.1293E+02	.1676E+01	.3183E+01	.4544E+01	.5621E+01	.6291E+01	.6666E+01	.6864E+01
47	$VOLUME = \sum A_i l_i$	.2475E+04	.1584E+04	.2467E+04	.2218E+04	.2087E+04	.2037E+04	.2017E+04	.2008E+04
48	$WT = (47)(\rho)$	.2475E+03	.1584E+03	.2467E+03	.2218E+03	.2087E+03	.2037E+03	.2017E+03	.2008E+03
49	$A_1^{k+1} = T_1/\bar{\sigma}_1 \; (32/\sigma_1)$	.1293E+02	.1582E+02	.1725E+02	.1869E+02	.1976E+02	.2043E+02	.2081E+02	.2101E+02
50	$A_2^{k+1} = T_2/\bar{\sigma}_2 \; (33/\sigma_2)$	.2929E+01	.1987E+01	.1403E+01	.8935E+00	.5126E+00	.2759E+00	.1433E+00	.1000E+00
51	$A_3^{k+1} = T_3/\bar{\sigma}_3 \; (34/\sigma_3)$	.1213E+01	.1676E+01	.3183E+01	.4544E+01	.5621E+01	.6291E+01	.6666E+01	.6864E+01

Note: If in rows 49-51 $A_i < A_{min}$ then $A_i = A_{min}$

Problem 3(b).

In this problem the relative design variable α_i is used. It is modified by using the relation

$$\alpha_i^{k+1} = \alpha_i^k \left(\frac{Q_{i1}}{\rho_i \ell_i \alpha_i^2} \right)_k^{1/2}$$

The Lagrange multipliers are estimated by using the relation (Equation 132)

$$\lambda_j^{k+1} = \lambda_j^k \left(\frac{\sigma_j}{\bar{\sigma}_j} \right)_k^{1/2} \qquad j=1,2,3$$

For the first iteration the λ_j's are assumed to be proportional to the forces in the bars. The relative design variables and the Lagrange multipliers are normalized after each iteration so that the maximum α_i or λ_j is equal to unity. A single virtual load vector is used with

$$f_x = \frac{E}{\ell_1} \cos\phi \cdot \lambda_1 \left(\frac{T_1}{ABS(T_1)} \right)$$

$$+\ 0$$

$$- \frac{E}{\ell_3} \cdot \cos\phi \cdot \lambda_3 \cdot \left(\frac{T_3}{ABS(T_3)} \right)$$

and

$$f_y = \frac{E}{\ell_1} \cdot \cos\theta \cdot \lambda_1 \left(\frac{T_1}{ABS(T_1)} \right)$$

$$+ \frac{E}{\ell_2} \cdot \lambda_2 \cdot \left(\frac{T_2}{ABS(T_2)} \right)$$

$$+ \frac{E}{\ell_3} \cdot \cos\theta \cdot \lambda_3 \cdot \left(\frac{T_3}{ABS(T_3)} \right)$$

The ratio $\frac{T_i}{ABS(T_i)}$ is equal to +1 or -1 depending on whether the force in the bar is Tension or Compression.

Row 48 shows the minimum weight of the structure to be 128.6 lb. This is achieved with five iterations. With additional iterations the weight increases. The cross-sectional areas for the minimum-weight design are $A_1 = 14.22$ in^2; $A_2 = 5.468$ in^2; $A_3 = 0.1003$ in^2. The Lagrange multipliers associated with this design are $\lambda_1 = 1.0$, $\lambda_2 = 0.086$ and $\lambda_3 = 0.01$. For convenience the flexibility coefficient is denoted by Qij instead of Rij.

Problem 3(b) Cont'd.

			CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	α_1		.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
2	α_2		.1000E+01	.9500E+00	.5781E+00	.4953E+00	.3844E+00	.3213E+00	.2842E+00	.2627E+00
3	α_3		.1000E+01	.7735E-02	.7085E-02	.7060E-02	.7050E-02	.7031E-02	.6112E-02	.4925E-02
4			~~.1000E+00~~	~~.1000E+09~~	~~.1000E+09~~	~~.1000E+09~~	~~.1000E+09~~	~~.1000E+09~~	~~.1000E+09~~	~~.1000E+09~~
5	E		.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08	.1000E+08
6	F_x		.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y		.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06
8	$\cos\theta$		.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$		.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$		.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$		.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta\cdot\cos\phi$		.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	l_1		.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	l_2		.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	l_3		.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/l_1		.1414E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06	.1414E+06
17	A_2E/l_2		.2000E+06	.1918E+06	.1156E+06	.9905E+05	.7689E+05	.6426E+05	.5684E+05	.5254E+05
18	A_3E/l_3		.1414E+06	.1094E+04	.1002E+04	.9985E+03	.9971E+03	.9943E+03	.8644E+03	.6965E+03
19	$\frac{A_1E}{l_1}+\frac{A_3E}{l_3}$ (16+17)		.2828E+06	.1425E+06	.1424E+06	.1424E+06	.1424E+06	.1424E+06	.1423E+06	.1421E+06
20	$\frac{A_1E}{l_1}-\frac{A_3E}{l_3}$ (16-17)		9.	.1403E+06	.1404E+06	.1404E+06	.1404E+06	.1404E+06	.1406E+06	.1407E+06

Problem 3(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.7126E+05	.7121E+05	.7121E+05	.7121E+05	.7121E+05	.7114E+05	.7106E+05
22	$K_2 = (12)(20)$	0.	.7016E+05	.7021E+05	.7021E+05	.7021E+05	.7021E+05	.7028E+05	.7036E+05
23	$K_3 = (10)(19) + 17$	.3414E+06	.2630E+06	.1960E+06	.1703E+06	.1481E+06	.1355E+06	.1280E+06	.1236E+06
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.1227E+11	-.4641E+10	-.2984E+10	-.7670E+09	.4962E+09	.1250E+10	.1712E+10
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4828E+11	-.1382E+11	-.8375E+10	-.7195E+10	-.5616E+10	-.4716E+10	-.4166E+10	-.3832E+10
26	$U_x = {}^{24}/_{25}$	.7071E+00	.8878E+00	.5542E+00	.4147E+00	.1366E+00	-.1052E+00	-.3819E+00	-.4468E+00
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.2828E+11	-.7235E+10	-.7221E+10	-.7221E+10	-.7221E+10	-.7220E+10	-.7201E+10	-.7176E+10
28	$U_y = {}^{27}/_{25}$	.5858E+00	.5236E+00	.8622E+00	.1004E+01	.1286E+01	.1531E+01	.1728E+01	.1872E+01
29	$U_1 = (26)(9) + (28)(9)$	.9142E+00	.9908E+00	.1002E+01	.1003E+01	.1006E+01	.1008E+01	.1009E+01	.1008E+01
30	$U_2 = U_y \; (28)$	.5858E+00	.5236E+00	.8622E+00	.1004E+01	.1286E+01	.1531E+01	.1728E+01	.1872E+01
31	$U_3 = -(26)(9) + (28)(9)$	-.8579E-01	-.2576E+00	.2178E+00	.4164E+00	.8126E+00	.1157E+01	.1436E+01	.1640E+01
32	$T_1 = \frac{A_1 E}{\ell_1} U_1 \; (16)(29)$	.1293E+06	.1411E+06	.1416E+06	.1418E+06	.1422E+06	.1426E+06	.1427E+06	.1426E+06
33	$T_2 = \frac{A_2 E}{\ell_2} U_2 \; (17)(30)$	.1172E+06	.1004E+06	.9969E+05	.9941E+05	.9885E+05	.9837E+05	.9825E+05	.9830E+05
34	$T_3 = \frac{A_3 E}{\ell_3} U_3 \; (18)(31)$	-.1213E+05	-.2017E+03	.2103E+03	.4158E+03	.8102E+03	.1150E+04	.1241E+04	.1142E+04
35	$\sigma_1 = \frac{T_1}{A_1} = {}^{32}/_{1}$	.1293E+06	.1411E+06	.1416E+06	.1418E+06	.1422E+06	.1426E+06	.1427E+06	.1426E+06
36	$\sigma_2 = {}^{T_2}/_{A_2} = {}^{33}/_{2}$	.1172E+06	.1047E+06	.1724E+06	.2007E+06	.2571E+06	.3062E+06	.3457E+06	.3745E+06
37	$\sigma_3 = {}^{T_3}/_{A_3} = {}^{34}/_{3}$	-.1213E+05	-.3643E+05	.3080E+05	.5889E+05	.1149E+06	.1636E+06	.2030E+06	.2319E+06
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\bar{\sigma}}$	.1293E+02	.1411E+02	.1416E+02	.1418E+02	.1422E+02	.1426E+02	.1427E+02	.1426E+02
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\bar{\sigma}}$	.2929E+01	.2618E+01	.4311E+01	.5018E+01	.6429E+01	.7655E+01	.8642E+01	.9362E+01
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\bar{\sigma}}$	.1213E+01	.3643E+01	.3080E+01	.5889E+01	.1149E+02	.1636E+02	.2030E+02	.2319E+02

Problem 3(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(U_x) = \frac{U_x}{\varepsilon_1} = \frac{26}{4}$	.7071E-00	.6070E-00	.5542E-00	.4147E-00	.1366E-00	.1052E-00	.3019E-00	.4460E-00
42	$\Lambda(U_y) = \frac{U_y}{\varepsilon_1} = \frac{28}{4}$	.5050E-00	.5236E-00	.8622E-00	.1004E-07	.1206E-07	.1531E-07	.1720E-07	.1072E-07
43	$\Delta(max)_{(38-42)}$	.1293E+02	.1411E+02	.1416E+02	.1418E+02	.1422E+02	.1636E+02	.2030E+02	.2319E+02
44	$A_1^* = \Lambda \cdot A_{1(43)(1)}$	.1293E+02	.1411E+02	.1416E+02	.1418E+02	.1422E+02	.1636E+02	.2030E+02	.2319E+02
45	$A_2^* = \Lambda \cdot A_{2(43)(2)}$	.1293E+02	.1353E+02	.8100E+01	.7025E+01	.5460E+01	.5257E+01	.5770E+01	.6093E+01
46	$A_3^* = \Lambda \cdot A_{3(43)(3)}$	.1293E+02	.1092E+00	.1004E+00	.1001E+00	.1003E+00	.1150E+00	.1241E+00	.1142E+00
47	$VOLUME = \sum A_i l_i$	.2479E+04	.1682E+04	.1418E+04	.1361E+04	.1286E+04	.1428E+04	.1733E+04	.1953E+04
48	$WT = (47)(\rho)$	.2479E+03	.1682E+03	.1418E+03	.1361E+03	.1286E+03	.1428E+03	.1733E+03	.1953E+03
49	$\lambda_1^{k+1} = \lambda_1^k (38)^{1/2}$	.1000E+01	.3757E+01	.3764E+01	.3766E+01	.3771E+01	.3776E+01	.3777E+01	.3776E+01
50	$\lambda_2^{k+1} = \lambda_2^k (39)^{1/2}$	.9062E+00	.1466E+01	.8103E+00	.4023E+00	.3247E+00	.2302E+00	.1055E+00	.1502E+00
51	$\lambda_3^{k+1} = \lambda_3^k (40)^{1/2}$	.9303E-01	.1791E+00	.8367E-01	.5395E-01	.4856E-01	.5288E-01	.6215E-01	.7924E-01
52	$\lambda_1^{k+1} = 49/Max(49-51)$	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
53	$\lambda_2^{k+1} = 50/Max(49-51)$	.9062E+00	.3903E+00	.2153E+00	.1201E+00	.8610E-01	.6309E-01	.4910E-01	.3979E-01
54	$\lambda_3^{k+1} = 51/Max(49-51)$	.9303E-01	.4767E-01	.2223E-01	.1432E-01	.1288E-01	.1379E-01	.1645E-01	.2099E-01
55	$\frac{E}{\lambda_1} \cdot Cos\phi \cdot \lambda_1 \cdot (SIGN\text{-}F_1)$	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
56	0.0	0.	0.	0.	0.	0.	0.	0.	0.
57	$-\frac{E}{I_3} \cdot Cos\phi \cdot \lambda_3 (SIGN\text{-}F_3)$	.9304E+04	.4767E+04	-.2223E+04	-.1432E+04	-.1288E+04	-.1379E+04	-.1646E+04	-.2099E+04
58	$\frac{E}{I_1} \cdot Cos\theta \cdot \lambda_1 \cdot (SIGN\text{-}F_1)$	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
59	$\frac{E}{I_2} \cdot \lambda_2 \cdot (SIGN\text{-}F_2)$	.1012E+06	.7805E+05	.4306E+05	.2561E+05	.1722E+05	.1262E+05	.9820E+04	.7958E+04
60	$\frac{E}{I_3} \cdot Cos\theta \cdot \lambda_3 \cdot (SIGN\text{-}F_3)$	-.9304E+04	-.4767E+04	.2223E+04	.1432E+04	.1288E+04	.1379E+04	.1646E+04	.2099E+04

Problem 3(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$f_x' = 55+56+57$	.1094E+06	.1040E+06	.9778E+05	.9857E+05	.9871E+05	.9862E+05	.9836E+05	.9790E+05
62	$f_y' = 58+59+60$	.2718E+06	.1733E+06	.1453E+06	.1270E+06	.1185E+06	.1140E+06	.1115E+06	.1101E+06
63	$K_2 \cdot f_y' - K_3 f_x'$ $(22)(62)-(23)(61)$	-.3735E+11	-.1540E+11	-.8068E+10	-.7862E+10	-.6298E+10	-.5356E+10	-.4754E+10	-.4357E+10
64	$u_x = 63/25$	.7735E+00	.1114E+01	.9633E+00	.1093E+01	.1121E+01	.1136E+01	.1141E+01	.1137E+01
65	$K_2 \cdot f_x' - K_1 \cdot f_y'$ $(22)(61)-(21)(62)$	-.3045E+11	-.4997E+10	-.3481E+10	-.2126E+10	-.1508E+10	-.1193E+10	-.1018E+10	-.9320E+09
66	$u_y = 65/25$	.7962E+00	.3616E+00	.4156E+00	.2955E+00	.2685E+00	.2529E+00	.2443E+00	.2432E+00
67	$u_1 = (64)(9)+(66)(8)$	.1110E+01	.1044E+01	.9750E+00	.9817E+00	.9829E+00	.9818E+00	.9797E+00	.9759E+00
68	$u_2 = u_y\ (66)$	.7962E+00	.3616E+00	.4156E+00	.2955E+00	.2685E+00	.2529E+00	.2443E+00	.2432E+00
69	$u_3 = (-64)(9)+(66)(8)$	.1618E-01	-.5322E+00	-.3873E+00	-.5638E+00	-.6031E+00	-.6241E+00	-.6342E+00	-.6320E+00
70	$t_1' = \frac{A_1 u_1 E}{l_1}(16)(67)$	.1570E+06	.1476E+06	.1379E+06	.1388E+06	.1390E+06	.1389E+06	.1385E+06	.1380E+06
71	$t_2' = \frac{A_2 u_2 E}{l_2}(17)(68)$	.1592E+06	.6934E+05	.4806E+05	.2927E+05	.2864E+05	.1625E+05	.1389E+05	.1278E+05
72	$t_3' = \frac{A_3 u_3 E}{l_3}(18)(69)$	.2276E+04	-.5021E+03	-.3808E+03	-.5629E+03	-.6014E+03	-.6206E+03	-.5402E+03	-.4402E+03
73	$Q_{11} = T_1 t_1' l_1/E$	.1439E+06	.1473E+06	.1381E+06	.1392E+06	.1398E+06	.1408E+06	.1398E+06	.1391E+06
74	$Q_{21} = T_2 t_2' l_2/E$	.9328E+05	.3401E+05	.2395E+05	.1455E+05	.1020E+05	.7994E+04	.6821E+04	.6286E+04
75	$Q_{31} = T_3 t_3' l_3/E$	-.1953E+03	.1160E+01	-.5988E+00	-.1655E+01	-.3445E+01	-.5048E+01	-.4018E+01	-.3556E+01
76	$Q_{11}/\alpha_1^2 \cdot \rho \cdot l_1$	.2029E+05	.2003E+05	.1953E+05	.1969E+05	.1977E+05	.1988E+05	.1977E+05	.1968E+05
77	$Q_{21}/\alpha_2^2 \cdot \rho \cdot l_2$	.1066E+05	.7573E+04	.1433E+05	.1186E+05	.1381E+05	.1549E+05	.1689E+05	.1821E+05
78	$Q_{31}/\alpha_3^2 \cdot \rho \cdot l_3$	-.2762E+02	.2742E+04	-.1687E+04	-.4695E+04	-.9802E+04	-.1444E+05	-.1821E+05	-.2073E+05
79	$\alpha_1^{k+1} = \alpha_1 (76)^{1/2}$	.1429E+03	.1443E+03	.1398E+03	.1403E+03	.1406E+03	.1407E+03	.1406E+03	.1403E+03
80	$\alpha_2^{k+1} = \alpha_2 (77)^{1/2}$	.1366E+03	.8344E+02	.6921E+02	.5395E+02	.4518E+02	.3999E+02	.3694E+02	.3546E+02

Problem 3(b) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
81	$\alpha_3^{k+1} = \alpha_3(78)^{1/2}$	0.	.4850E+00	0.	0.	0.	0.	0.	0.
82	$\alpha_1 = 79/Max(79-81)$	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
83	$\alpha_2 = 80/Max(79-81)$	.9500E+00	.5781E+00	.4953E+00	.3844E+00	.3213E+00	.2842E+00	.2627E+00	.2520E+00
84	$\alpha_3 = 81/Max(79-81)$	0.	.2806E-02	0.	0.	0.	0.	0.	0.
85	α_1 checked	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
86	α_2 for λ_{min}	.9500E+00	.5781E+00	.4953E+00	.3844E+00	.3213E+00	.2842E+00	.2627E+00	.2520E+00
87	α_3 $\alpha_{min} = \frac{\lambda_{min}}{\Delta(max)}$ (43)	.7739E-02	.7805E-02	.7860E-02	.7850E-02	.7831E-02	.6112E-02	.4925E-02	.4312E-02

Problem 3(c).

In this problem the design variables are modified by using the relation (Equation 131)

$$A_i^{k+1} = A_i^k \left(1 + \frac{1}{r} \left(\sum_{j=1}^{2} \lambda_j \frac{Q_{ij}}{\rho_i \ell_i A_i^2} - 1 \right) \right)_k$$

with step size parameter $r = 2$. The equations used for evaluating the Lagrange multipliers are the same as those of Problem 2(a). The stress constraints in elements 1 and 3 are assumed to be potentially active in order to reduce the size of the problem solution. The stress in element 2 is inactive, and assuming it to be potentially active would only increase the size of the problem. For element 1 the virtual load is

$$f_x^1 = \frac{E}{\ell_1} \cdot \cos\phi \cdot \left(\frac{T_1}{ABS(T_1)} \right)$$

$$f_y^1 = \frac{E}{\ell_1} \cdot \cos\theta \cdot \left(\frac{T_1}{ABS(T_1)} \right)$$

and for element 3 the virtual load is

$$f_x^2 = - \frac{E}{\ell_3} \cdot \cos\phi \cdot \left(\frac{T_3}{ABS(T_3)} \right)$$

$$f_y^2 = + \frac{E}{\ell_3} \cdot \cos\theta \cdot \left(\frac{T_3}{ABS(T_3)} \right)$$

The flexibility coefficients are denoted by Qij instead of Rij for convenience. The weight of the optimum design is 126.06 lb, and the cross-sectional areas of the three members are $A_1 = 14.24$ in^2, $A_2 = 4.929$ in^2 and $A_3 = 0.1$ in^2. The stresses in the three elements are 10,000 psi, 20,000 psi and 10,000 psi respectively. The stress constraint in element 1 is active for all iterations, but the stress constraint in

element 3 becomes active only after the fourth iteration. The stress in element 2 is not active.

Comparing this design with the one obtained by using the FSD algorithm, one finds that the two designs are not the same although the stress distribution for them is the same. The design obtained by the FSD algorithm is non-optimum.

Problem 3(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
1	A_1	.1000E+01	.1400E+02	.1405E+02	.1517E+02	.1502E+02	.1203E+02	.1428E+02	.1423E+02
2	A_2	.1000E+01	.1051E+02	.7952E+01	.5700E+01	.4047E+01	.4731E+01	.4050E+01	.4926E+01
3	A_3	.1000E+01	.6760E+01	.3404E+01	.1490E+01	.4377E+00	.2936E+00	.1000E+00	.1000E+00
4		~~.1000E+14~~	~~.1000E+14~~	~~.1000E+14~~	~~.1000E+14~~	~~.1000E+14~~	~~.1000E+14~~	~~.1000E+14~~	~~.1000E+14~~
5	E	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00	.1000E+00
6	F_x	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
7	F_y	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06	.2000E+06
8	$\cos\theta$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
9	$\cos\phi$	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00	.7071E+00
10	$\cos^2\theta$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
11	$\cos^2\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
12	$\cos\theta \cdot \cos\phi$	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00	.5000E+00
13	l_1	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
14	l_2	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02	.5000E+02
15	l_3	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02	.7071E+02
16	A_1E/l_1	.1414E+06	.1991E+07	.2100E+07	.2145E+07	.2125E+07	.1814E+07	.2020E+07	.2012E+07
17	A_2E/l_2	.2000E+06	.2101E+07	.1590E+07	.1156E+07	.8095E+06	.9462E+06	.9716E+06	.9852E+06
18	A_3E/l_3	.1414E+06	.9561E+06	.4013E+06	.2118E+06	.6190E+05	.4153E+05	.1414E+05	.1414E+05
19	$\frac{A_1E}{l_1} + \frac{A_3E}{l_3}$ (16+17)	.2028E+06	.2947E+07	.2502E+07	.2357E+07	.2187E+07	.1855E+07	.2034E+07	.2026E+07
20	$\frac{A_1E}{l_1} - \frac{A_3E}{l_3}$ (16-17)	0.	.1035E+07	.1619E+07	.1933E+07	.2063E+07	.1772E+07	.2006E+07	.1998E+07

Problem 3(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
21	$K_1 = (11)(19)$	.1414E+06	.1474E+07	.1291E+07	.1178E+07	.1093E+07	.9277E+06	.1017E+07	.1013E+07
22	$K_2 = (12)(20)$	0.	.5175E+06	.8895E+06	.9664E+06	.1031E+07	.8862E+06	.1003E+07	.9991E+06
23	$K_3 = (10)(19) + 17$	.3414E+06	.3575E+07	.2881E+07	.2334E+07	.1903E+07	.1874E+07	.1909E+07	.1998E+07
24	$K_2 \cdot F_y - K_3 \cdot F_x$ $(22)(7) - (23)(6)$	-.3414E+11	-.2548E+12	-.1262E+12	-.4013E+11	.1601E+11	-.1019E+11	.1783E+10	-.2812E+08
25	$K_2 \cdot K_2 - K_1 \cdot K_3$ $(22)(22) - (21)(23)$	-.4828E+11	-.5088E+13	-.3064E+13	-.1816E+13	-.1017E+13	-.9531E+12	-.1017E+13	-.1027E+13
26	$U_x = 24/25$	.7071E+00	.5008E-01	.4120E-01	.2218E-01	-.1575E-01	.1065E-01	-.1675E-02	.1968E-04
27	$K_2 \cdot F_x - K_1 \cdot F_y$ $(22)(6) - (21)(7)$	-.2828E+11	-.2438E+12	-.1772E+12	-.1398E+12	-.1155E+12	-.9692E+11	-.1031E+12	-.1027E+12
28	$U_y = 27/25$	.5858E+00	.4859E-01	.5784E-01	.7653E-01	.1136E+00	.1017E+00	.1014E+00	.1001E+00
29	$U_1 = (26)(9) + (28)(9)$	.9142E+00	.7828E-01	.7883E-01	.6974E-01	.6922E-01	.7944E-01	.7853E-01	.7877E-01
30	$U_2 = U_y$ (28)	.5858E+00	.4859E-01	.5784E-01	.7653E-01	.1136E+00	.1017E+00	.1014E+00	.1001E+00
31	$U_3 = -(26)(9) + (28)(9)$	-.8579E-01	-.1559E-02	.1177E-01	.3849E-01	.9158E-01	.6438E-01	.7298E-01	.7875E-01
32	$T_1 = \frac{A_1 E}{L_1} U_1$ (16)(29)	.1293E+06	.1399E+06	.1471E+06	.1496E+06	.1471E+06	.1441E+06	.1425E+06	.1424E+06
33	$T_2 = \frac{A_2 E}{L_2} U_2$ (17)(30)	.1172E+06	.1021E+06	.9199E+05	.8847E+05	.9199E+05	.9622E+05	.9854E+05	.9859E+05
34	$T_3 = \frac{A_3 E}{L_3} U_3$ (18)(31)	-.1213E+05	-.1490E+04	.5665E+04	.8155E+04	.5663E+04	.2673E+04	.1031E+04	.1001E+04
35	$\sigma_1 = \frac{T_1}{A_1} = 32/1$	.1293E+06	.9940E+04	.9983E+04	.9863E+04	.9798E+04	.1123E+05	.9974E+04	.1001E+05
36	$\sigma_2 = T_2/A_2 = 33/2$	.1172E+06	.9719E+04	.1157E+05	.1531E+05	.2273E+05	.2034E+05	.2020E+05	.2001E+05
37	$\sigma_3 = T_3/A_3 = 34/3$	-.1213E+05	-.2204E+03	.1664E+04	.5444E+04	.1294E+05	.9104E+04	.1031E+05	.1001E+05
38	$\Lambda(\sigma_1) = \frac{\sigma_1}{\sigma_1}$ (abs)	.1293E+02	.9940E+00	.9983E+00	.9863E+00	.9790E+00	.1123E+01	.9974E+00	.1001E+01
39	$\Lambda(\sigma_2) = \frac{\sigma_2}{\sigma_2}$ (abs)	.2929E+01	.2430E+00	.2892E+00	.3827E+00	.5682E+00	.5085E+00	.5071E+00	.5004E+00
40	$\Lambda(\sigma_3) = \frac{\sigma_3}{\sigma_3}$ (abs)	.1213E+01	.2204E-01	.1664E+00	.5444E+00	.1294E+01	.9104E+00	.1031E+01	.1001E+01

Problem 3(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
41	$\Lambda(U_x) = \frac{U_x}{\varepsilon_1} = \frac{26}{4}$	.7071E-13	.5000E-14	.4120E-14	.2218E-14	-.1575E-14	.1065E-14	-.1675E-15	.1960E-17
42	$\Lambda(U_y) = \frac{U_y}{\varepsilon_1} = \frac{28}{4}$	.5050E-13	.4059E-14	.5784E-14	.7653E-14	.1136E-13	.1017E-13	.1014E-13	.1001E-13
43	$\Delta(max)\ (38-41)$	.1293E+02	.9940E+00	.9903E+00	.9863E+00	.1294E+01	.1123E+01	.1031E+01	.1001E+01
44	$A_1^* = \Lambda \cdot A_1\ (43)(1)$	.1293E+02	.1399E+02	.1471E+02	.1496E+02	.1944E+02	.1441E+02	.1472E+02	.1424E+02
45	$A_2^* = \Lambda \cdot A_2\ (43)(2)$	.1293E+02	.1044E+02	.7875E+01	.5781E+01	.5237E+01	.5315E+01	.5089E+01	.4930E+01
46	$A_3^* = \Lambda \cdot A_3\ (43)(3)$	.1293E+02	.6720E+01	.3371E+01	.1477E+01	.5663E+00	.3299E+00	.1031E+00	.1001E+00
47	$VOLUME = \Sigma A_i l_i$	.2475E+04	.1987E+04	.1672E+04	.1447E+04	.1677E+04	.1300E+04	.1299E+04	.1261E+04
48	$WT = (47)(\rho)$	.2475E+03	.1987E+03	.1672E+03	.1447E+03	.1677E+03	.1300E+03	.1299E+03	.1261E+03
49	$f_x' = \frac{E}{l_1} \cdot Cos\phi \cdot (sig-T_1)$	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
50	$f_y' = \frac{E}{l_1} Cos\theta \cdot (sig-T_1)$	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
51	$K_2 \cdot f_y' - K_3 \cdot f_x'$ $(22)(50)-(23)(49)$	-.3414E+11	-.3057E+12	-.2072E+12	-.1368E+12	-.8714E+11	-.9877E+11	-.9858E+11	-.9993E+11
52	$u_x' = 51/25$	.7071E+00	.6115E-01	.6762E-01	.7538E-01	.8572E-01	.1036E+00	.9696E-01	.9734E-01
53	$K_2 \cdot f_x' - K_1 \cdot f_y'$ $(23)(49)-(21)(50)$	-.1414E+11	-.9561E+11	-.4813E+11	-.2118E+11	-.6190E+10	-.4153E+10	-.1414E+10	-.1414E+10
54	$u_y' = 53/25$	.2929E+00	.1912E-01	.1571E-01	.1166E-01	.6089E-02	.4357E-02	.1391E-02	.1377E-02
55	$u_1' = (52)(9)+(54)(8)$	.7071E+00	.9676E-01	.5092E-01	.6150E-01	.6492E-01	.7636E-01	.6955E-01	.6980E-01
56	$u_2' = u_y\ (54)$	.2929E+00	.1912E-01	.1571E-01	.1166E-01	.6089E-02	.4357E-02	.1391E-02	.1377E-02
57	$u_3' = -(52)(9)+(54)(8)$	-.2929E+00	-.2972E-01	-.3670E-01	-.4500E-01	-.5631E-01	-.7020E-01	-.6758E-01	-.6785E-01
58	$t_1' = \frac{A_1 u_1' E}{l_1}\ (16)(55)$	.1000E+06	.1130E+06	.1230E+06	.1319E+06	.1379E+06	.1385E+06	.1405E+06	.1405E+06
59	$t_2' = \frac{A_2 u_2' E}{l_2}\ (17)(56)$	.5050E+05	.4010E+05	.2490E+05	.1340E+05	.4929E+04	.4123E+04	.1352E+04	.1357E+04
60	$t_3' = \frac{A_3 u_3' E}{l_3}\ (18)(57)$	-.4142E+05	-.2841E+05	-.1767E+05	-.9533E+04	-.3405E+04	-.2915E+04	-.9557E+03	-.9596E+03

Problem 3(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
61	$Q_{11} = \frac{T_1 t_1' l_1}{E}$	.9142E+05	.1118E+06	.1287E+06	.1395E+06	.1435E+06	.1411E+06	.1415E+06	.1415E+06
62	$Q_{21} = \frac{T_2 t_2' l_2}{E}$	.3431E+05	.2051E+05	.1149E+05	.5964E+04	.2267E+04	.1983E+04	.6659E+03	.6689E+03
63	$Q_{31} = \frac{T_3 t_3' l_3}{E}$	.3553E+04	.2994E+03	-.7077E+03	-.5497E+03	-.1396E+03	-.5511E+02	-.6967E+01	-.6789E+01
64	Passive Elements	0.	0.	0.	0.	0.	0.	3	3
65	$C_1^* = \Sigma \frac{Q \lambda_1}{A_1} (P)$	0.	0.	0.	0.	0.	0.	-.6758E+02	-.6783E+02
66	$f_x^c = -\frac{E}{l_3} \cdot \cos\phi (\sin - T_3)$	.1000E+06	.1000E+06	-.1000E+06	-.1000E+06	-.1000E+06	-.1000E+06	-.1000E+06	-.1000E+06
67	$f_y^c = +\frac{E}{l_3} \cdot \cos\theta (\sin - T_3)$	-.1000E+06	-.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06	.1000E+06
68	$K_2 \cdot f_y^c - K_3 \cdot f_x^c$ (22)(67) - (23)(66)	-.3414E+11	-.4092E+12	.3691E+12	.3301E+12	.2934E+12	.2760E+12	.2991E+12	.2998E+12
69	$u_x^c = 68/25$	.7071E+00	.8105E-01	-.1205E+00	-.1617E+00	-.2087E+00	-.2096E+00	-.2942E+00	-.2920E+00
70	$K_2 \cdot f_x^c - K_1 \cdot f_y^c$ (23)(66) - (21)(67)	.1414E+11	.1991E+12	-.2108E+12	-.2145E+12	-.2125E+12	-.1814E+12	-.2020E+12	-.2012E+12
71	$u_y^c = 70/25$	-.2929E+00	-.3902E-01	.6855E-01	.1181E+00	.2090E+00	.1903E+00	.1987E+00	.1960E+00
72	$u_1^c = (69)(9) + (71)(8)$	.2929E+00	.2972E-01	-.3670E-01	-.4500E-01	-.5631E-01	-.7020E-01	-.6758E-01	-.6785E-01
73	$u_2^c = u_y^c$	-.2929E+00	-.3902E-01	.6855E-01	.1181E+00	.2090E+00	.1903E+00	.1987E+00	.1960E+00
74	$u_3^c = -(69)(9) + (71)(8)$	-.7071E+00	-.8684E-01	.1336E+00	.2120E+00	.3519E+00	.3394E+00	.3485E+00	.3451E+00
75	$t_1^2 = \frac{A_1 u_1^c E}{l_1} (16)(72)$	.4142E+05	.5917E+05	-.7789E+05	-.9651E+05	-.1196E+06	-.1273E+06	-.1365E+06	-.1365E+06
76	$t_2^2 = \frac{A_2 u_2^c E}{l_2} (17)(73)$	-.5858E+05	-.8367E+05	.1090E+06	.1365E+06	.1692E+06	.1801E+06	.1930E+06	.1931E+06
77	$t_3^2 = \frac{A_3 u_3^c E}{l_3} (18)(74)$	-.1000E+06	-.8226E+05	.6433E+05	.4491E+05	.2178E+05	.1409E+05	.4929E+04	.4880E+04
78	$Q_{12} = \frac{T_1 t_1^2 l_1}{E}$	.3787E+05	.5854E+05	-.8018E+05	-.1021E+06	-.1244E+06	-.1297E+06	-.1375E+06	-.1375E+06
79	$Q_{22} = \frac{T_2 t_2^2 l_2}{E}$	-.3431E+05	-.4272E+05	.5015E+05	.6038E+05	.7782E+05	.8663E+05	.9511E+05	.9518E+05
80	$Q_{32} = T_3 t_3^2 l_3 / E$	.8579E+04	.8668E+03	.2577E+04	.2589E+04	.8723E+03	.2664E 33	.3593E+02	.3452E+C2

Problem 3(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
81	$C_2^* = \Sigma \frac{Q_{i2}}{\lambda_i}(P)$	0.	0.	0.	0.	0.	0.	.3485E+03	.3449E+03
82	$B_1 = \Sigma \frac{Q_{i1}}{A_i}(\text{Actin})^2$	.1000E+05	.1000E+05	.1000E+05	.1000E+05	.7566E+04	.1000E+05	.9743E+04	.1007E+05
83	$B_2 = \Sigma \frac{Q_{i2}}{A_i}(\text{Actin})^2$	.9304E+03	.2210E+03	.1681E+04	.5519E+04	.1000E+05	.8183E+04	.9651E+04	.9651E+04
84	$R_1 = 2(C_1^* + B_1 - \bar{\sigma}_1) + B_1$	.1000E+05	.1000E+05	.1000E+05	.1000E+05	.2697E+04	.1000E+05	.9093E+04	.1007E+05
85	$R_2 = 2(C_2^* + B_2 - \bar{\sigma}_2) + B_2$	-.1710E+05	-.1933E+05	-.1496E+05	-.3442E+04	.1000E+05	.4310E+04	.9651E+04	.9643E+04
86	$A = \Sigma \frac{Q_{i1} Q_{i1}}{\rho_i \lambda_i A_i^3}(\text{Actin})$	.6567E+06	.7193E+06	.7922E+06	.8740E+06	.4185E+06	.9586E+06	.8877E+06	.9803E+06
87	$B = \Sigma \frac{Q_{i1} Q_{i2}}{\rho_i \lambda_i A_i^3}(\text{Actin})$	.1196E+06	.1041E+06	-.2294E+06	-.2754E+06	-.1927E+06	-.6944E+06	-.7611E+06	-.8460E+06
88	$C = \Sigma \frac{Q_{i2} Q_{i2}}{\rho_i \lambda_i A_i^3}(\text{Actin})$	.2076E+06	.4978E+06	.1340E+07	.4678E+07	.9324E+07	.1107E+08	.1524E+08	.1605E+08
89	$B \cdot B - A \cdot B$ (87)(87) - (86)(89)	-.1221E+12	-.3242E+12	-.1809E+13	-.4086E+13	-.3864E+13	-.1013E+14	-.1295E+14	-.1501E+14
90	$B \cdot R_2 - C \cdot R_1$ (87)(85) - (88)(84)	-.4131E+10	-.8537E+10	-.9967E+10	-.4575E+11	-.2788E+11	-.1137E+12	-.1459E+12	-.1697E+12
91	$\lambda_1 = 90/89$	.3385E-01	.2633E-01	.9880E-02	.1142E-01	.7887E-02	.1122E-01	.1127E-01	.1130E-01
92	$B \cdot R_1 - A \cdot R_2$ (87)(84) - (86)(85)	.1240E+11	.1575E+11	.9556E+10	.2535E+09	-.4784E+10	-.1188E+11	-.1549E+11	-.1797E+11
93	$\lambda_2 = 92/89$	-.1023E+00	-.4858E-01	-.9472E-02	-.6338E-04	.1217E-02	.1093E-02	.1196E-02	.1197E-02
94	IF (93) < 0 $\lambda_1 = R_1/A$	.1523E-01	.1390E-01	.1262E-01	.1144E-01	.7887E-02	.1122E-01	.1127E-01	.1130E-01
95	IF (93) < 0 $\lambda_2 = 0.0$	0.	0.	0.	0.	.1217E-02	.1093E-02	.1196E-02	.1197E-02
96	$\lambda_1 Q_{11} + \lambda_2 Q_{12}$ (94)(61) + (95)(78)	.1392E+04	.1555E+04	.1625E+04	.1596E+04	.8537E+03	.1442E+04	.1430E+04	.1434E+04
97	$\lambda_1 Q_{21} + \lambda_2 Q_{22}$ (94)(62) + (95)(79)	.5225E+03	.2852E+03	.1451E+03	.6824E+02	.1186E+03	.1170E+03	.1213E+03	.1215E+03
98	$\lambda_1 Q_{31} + \lambda_2 Q_{32}$ (94)(63) + (95)(80)	.5411E+02	.4162E+01	-.8933E+01	-.6290E+01	.8393E-01	-.3273E+00	-.3553E-01	-.3541E-01
99	$96/(\lambda_1 \cdot A_1^{*2} \cdot \rho)$	.1178E+01	.1123E+01	.1062E+01	.1009E+01	.3195E+00	.9823E+00	.9329E+00	.1000E+01
100	$97/(\lambda_2 \cdot A_2^{*2} \cdot \rho)$	.6252E+00	.5231E+00	.4678E+00	.4200E+00	.8067E+00	.8281E+00	.9670E+00	.9996E+00

Problem 3(c) Cont'd.

		CYCLE 1	CYCLE 2	CYCLE 3	CYCLE 4	CYCLE 5	CYCLE 6	CYCLE 7	CYCLE 8
101	$98/(\ell_3 \cdot A_3^{*2} \cdot \rho)$	.4570E-01	.1304E-01	-.1112E+00	-.4075E+00	.3701E-01	-.4253E+00	-.4727E+00	-.4999E+00
102	$A_1^{k+1} = A_1^k(1+0.5(\underline{99}-1))$	.1400E+02	.1485E+02	.1517E+02	.1502E+02	.1203E+02	.1420E+02	.1423E+02	.1424E+02
103	$A_2^{k+1} = A_2^k(1+0.5(100-1))$	.1051E+02	.7952E+01	.5700E+01	.4047E+01	.4731E+01	.4858E+01	.4926E+01	.4929E+01
104	$A_3^{k+1} = A_3^k(1+0.5(\underline{101}-1))$	.6760E+01	.3404E+01	.1490E+01	.4377E+00	.2936E+00	.9408E-01	.2710E-01	.2503E-01
105	A_1 Checked for Amin	.1400E+02	.1485E+02	.1517E+02	.1502E+02	.1203E+02	.1420E+02	.1423E+02	.1424E+02
106	A_2	.1051E+02	.7952E+01	.5700E+01	.4047E+01	.4731E+01	.4858E+01	.4926E+01	.4929E+01
107	A_3	.6760E+01	.3404E+01	.1490E+01	.4377E+00	.2936E+00	.1000E+00	.1000E+00	.1000E+00

REFERENCES

1. V. B. Venkayya, N. S. Khot and L. Berke, "Application of Optimality Criteria Approaches to Automated Design of large Practical Structures," Second Symp. Struct. Opt., AGARD-CP-123, Milan, Italy (1973).

2. L. Berke and N. S. Khot, "Use of Optimality Criteria Methods for Large Scale Systems," AGARD Lecture Series No. 70, Struct. Opt. (1974).

3. N. S. Khot, V. B. Venkayya and L. Berke, "Experience with Minimum Weight Design of Structures Using Optimality Criteria Methods," 2nd Int. Conf. Vehicle Struct. Mech., Southfield, Michigan (1977).

4. J. Kiusalaas, "Minimum Weight Design of Structures via Optimality Criteria," NASA TN D-7115 (1972).

5. V. B. Venkayya, N. S. Khot and V. S. Reddy, "Energy Distribution in an Optimum Structural Design," AFFDL-TR-68-156 (1969).

6. L. Berke, "An Efficient Approach to the Minimum Weight Design of Deflection Limited Structures," AFFDL-TM-70-4-FDTR (1970).

7. V. B. Venkayya, "Design of Optimum Structures," J. Computers and Struct. 1, 265-309 (1971).

8. I. C. Taig and R. I. Kerr, "Optimization of Aircraft Structures with Multiple Stiffness Requirements," Second Symp. Struct. Opt., AGARD-CP-123, Milan, Italy (1973).

9. R. A. Gellatly and L. Berke, "Optimality-Criterion Algorithms," Optimum Structural Design, edited by R. M. Gallagher and O. C. Zienkiewicz, John Wiley & Sons, London (1973).

10. J. C. Nagteggal, "A New Approach to Optimal Design of Elastic Structures," Computational Methods in Applied Mechanics and Engineering, Vol. 2, pp. 255-264, Feb (1973).

11. N. S. Khot, V. B. Venkayya, C. D. Johnson and V. A. Tischler, "Optimization of Fiber Reinforced Composite Structures," Int. J. Solids Structures, Vol. 9, pp. 1225-1236, Pergamon Press, Sept (1973).

12. N. S. Khot, V. B. Venkayya and L. Berke, "Optimum Design of Composite Structures with Stress and Deflection Constraints," AIAA Paper No. 75-141, presented at AIAA 13th Aerospace Sciences Meeting, Pasadena, California (1975).

13. R. A. Gellatly and L. Berke, "Optimal Structural Design," AFFDL-TR-70-165, Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio (1975).

REFERENCES (Cont'd)

14. D. Rizzi, "Optimization of Multi-Constrained Structures Based on Optimality Criteria," Proc. AIAA/ASME/SAE 17th Struct., Structural Dynamics and Materials Conf., King of Prussia, Pennsylvania, 448-462 (1976).

15. M. W. Dobbs and R. B. Nelson, "Application of Optimality Criteria to Automated Structural Design," AIAA J. 14, 1436-1443 (1976).

16. S. A. Segenreich, N. A. Zouain and J. Herskovits, "An Optimality Criteria Method Based on Slack Variables Concept for Large Scale Structural Optimization," Proc. Symp. Applications of Computer Meth. in Engng (Ed. C. Wellford, Jr.), University of Southern California, 563-575 (1977).

17. F. Austin, "A Rapid Optimization Procedure for Structures Subjected to Multiple Constraints," Paper No. 77-374, AIAA/ASME/SAE 18th Struct., Structural Dynamics and Materials Conf., San Diego, California (1977).

18. G. V. Rao, C. P. Shore and R. Narayanaswami, "An Optimality Criterion for Resizing Heated Structures with Temperature Constraints," NASA TN D-8525 (1977).

19. M. R. Khan, K. D. Willmert and W. A. Thornton, "A New Optimality Criterion Method for Large Scale Structures," AIAA/ASME 19th SDM Conf., Bethesda, Md., 47-58 (1978).

20. L. Berke and N. S. Khot, "A Simple Virtual Strain Energy Method to Fully Stress Design Structures with Dissimilar Stress Allowables and Material Properties," AFFDL-TM-77-28-FBR.

21. J. Kiusalaas, "Optimum Design of Structures with Buckling Constraints," Int. J. Solids and Struct. 9, 863-878 (1973).

22. N. S. Khot, V. B. Venkayya and L. Berke, "Optimum Structural Design with Stability Constraints," Int. J. Num. Math. Engng. 10, 1097-1114 (1976).

23. V. B. Venkayya, N. S. Khot, V. A. Tischler and R. F. Taylor, "Design of Optimum Structures for Dynamic Loads," Third Conf. Matrix Meth. Struct. Mech., Wright-Patterson AFB, Ohio, 619-658 (1971).

24. V. B. Venkayya and N. S. Khot, "Design of Optimum Structures to Impulse Type Loading," AIAA J. 13, 989-994 (1975).

25. J. Kiusalaas and R. Shaw, "An Algorithm for Optimal Design with Frequency Constraints," Int. J. for Num. Meth. in Engg., Vol. 13, No. 2, Det (1978).

REFERENCES (Concluded)

26. V. B. Venkayya, "Structural Optimization: A Review and Some Recommendations," Int. J. for Num. Meth. in Engg., Vol. 13, No. 2, Dec (1978).

27. N. S. Khot, "Optimization of Aerospace Structures to Satisfy Strength, Displacement, and Stability Requirements - A Review and Assessment of the State-of-the-Art," AFFDL-TM-79-81-FBR.

28. N. S. Khot, L. Berke and V. B. Venkayya, "Comparisons of Optimality Criteria Algorithms for Minimum Weight Design of Structures," AIAA J. 17, 182-190 (1979).

29. C. Fleury and G. Sander, "Structural Optimization by Finite Element," L.T.A.S. Report SA-58, University of Leige, 1978.

30. N. S. Khot, L. Berke and V. B. Venkayya, "Minimum Weight Design of Structures by the Optimality Criterion and Projection Method," AIAA 79-0720, Paper presented at AIAA/ASME/ASCE/AHS 20th Structures, Structural Dynamics and material Conference, St. Louis, MO, April (1979).

31. N. S. Khot, "Algorithms Based on Optimality Criteria to Design Minimum Weight Structures," to be published in the Special Issue 'Optimum Design of Large Structural Systems, Engineering Optimization Journal,' (1980).

32. R. A. Gellatly, D. M. Dupree and L. Berke, "OPTIM II: A Magic Compatible Large Scale Automated Minimum Weight Design Program," AFFDL-TR-74-97, I and II (1974).

33. N. S. Khot, "Computer Program (OPTCOMP) for Optimization of Composite Structures for Minimum Weight Design," AFFDL-TR-76-149 (1977).

34. V. B. Venkayya and V. A. Tischler, "OPTSTAT - A Computer Program for Optimal Design of Structures Subjected to Static Loads," AFFDL-TR-79-(in preparation).

35. J. Kiusalaas and G. B. Reddy, "DESAP 2 - A Structural Design Program with Stress and Buckling Constraints," NASA CR-2797 to 2799 (3 volumes), National Aeronautics and Space Administration, Washington, D.C. (1977).

36. G. Isakson and H. Pardo, "ASOP-3: A Program for the Minimum-Weight Design of Structures Subjected to Strength and Deflection Constraints," AFFDL-TR-76-157.

37. N. S. Khot, V. B. Venkayya, L. Berke and K. Schrader, "Optimum Design of Composite Wing Structures with Twist Constraint for Aeroelastic Tailoring," AFFDL-TR-76-117, Dec (1976).

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